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Singularities in Free Boundaries

submitted by

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for the degree of Doctor of Philosophy

of the

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Department of Mathematical Sciences

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Summary

This thesis is an investigation of the properties of Bernoulli free boundaries, in particular those related to the presence of singularities.

A Bernoulli free-boundary problem is one of finding domains in the plane on which a harmonic function simultaneously satisfies linear homogeneous Dirichlet and non-homogeneous Neumann boundary conditions. The boundary of such a domain is called a free boundary because it is not prescribed a priori. The motivating example for the study of Bernoulli problems occurs in the theory of steady hydrodynamic waves.

In this thesis we show that, for a large class of Bernoulli problems, a free boundary which is symmetric with respect to a vertical line through an isolated singular point must necessarily have a corner at that point, and we give a formula for the contained angle. We also show that, even in the presence of singularities, a geometrically simple Bernoulli free boundary is necessarily symmetric.

Other results of the thesis refer to regularity and geometric properties of Bernoulli free boundaries, bounds satisfied by solutions, analytic continuation of solutions in the complex domain and, for the water-wave problem, the existence of singular solutions and Gibbs Phenomenon.

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Contents

Chapter 1

Introduction

1.1 Description of the Problem

The *Stokes-wave problem* is one of the classical problems of nonlinear hydrodynamics. In recent years there have been significant developments in the rigorous analysis of this problem, some of which are summarized in the survey paper [33] and the book [2]. Most recently, a substantial paper of Shargorodsky and Toland [27] has laid the foundations for extensions of these methods to a general class of geometric problems, called *Bernoulli problems*, in which the Stokes-wave problem appears as a member of a much wider class.

This thesis continues the programme initiated in [27], by addressing some questions concerning the properties of Bernoulli free boundaries, in particular those related to the presence of singularities.

We start with the definition of a Bernoulli free-boundary problem, as systematized by Shargorodsky and Toland [27]. Given a curve $\mathcal{S}$ in the (X, Y)-plane, where

$$\mathcal{S} := \{(u(s), v(s)) : s \in \mathbb{R}\}, \tag{1.1a}$$

$$(u, v) \text{ is injective and absolutely continuous,} \tag{1.1b}$$

$$u'(s)^2 + v'(s)^2 > 0 \text{ for almost all s,} \tag{1.1c}$$

$$s \mapsto (u(s) - s, v(s)) \text{ is } 2\pi\text{-periodic,} \tag{1.1d}$$

let Ω denote the domain below $\mathcal{S}$, and consider the problem of finding ψ with

$$\psi \in C^2(\Omega) \cap C(\overline{\Omega}), \tag{1.1e}$$

$$\Delta\psi = 0 \text{ in } \Omega, \tag{1.1f}$$

$$\psi \text{ is } 2\pi\text{-periodic in } X, \tag{1.1g}$$

$$\nabla\psi(X,Y) \to (0,1) \text{ as } Y \to -\infty, \text{ uniformly in } X, \tag{1.1h}$$

$$\psi = 0 \text{ on } \mathcal{S}. \tag{1.1i}$$

Classical theory ensures the existence of a unique solution which, as shown in [27], satisfies $\psi < 0$ in Ω. It follows that the normal derivative of ψ is non-negative at points on $\mathcal{S}$ where it exists.

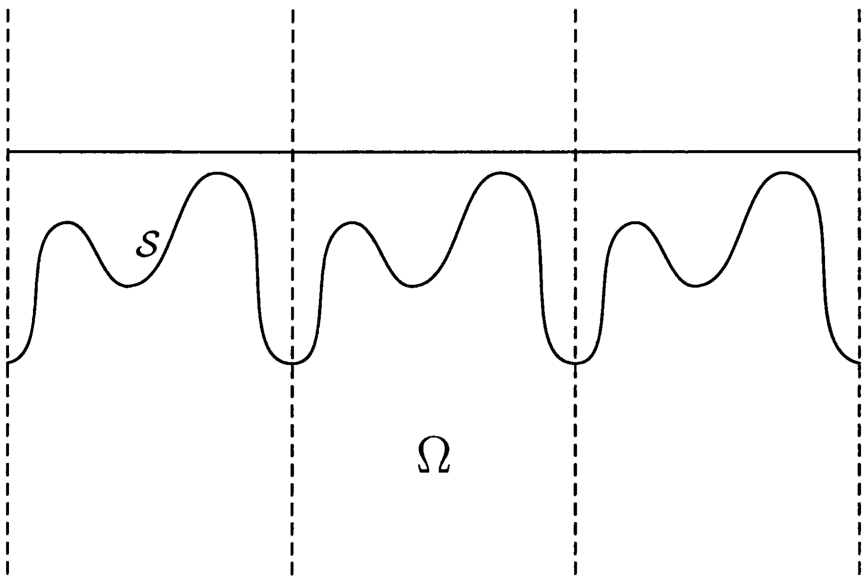

Figure 1-1: Possible profile of a free boundary

Let h be a given continuous function. A Bernoulli problem is one of determining those curves $\mathcal{S}$ with the additional property that the solution of this Dirichlet problem satisfies the inhomogeneous Neumann boundary condition

$$\frac{\partial \psi}{\partial n} = h(Y) \text{ almost everywhere on } \mathcal{S}. \tag{1.1j $'$}$$

Here n denotes the outward normal to Ω at points of $\mathcal{S}$. There is no loss of generality in assuming that $h \geq 0$. Note that, in the presence of (1.1i), (1.1j $'$) is formally equivalent to

$$|\nabla \psi|^2 = \lambda(Y) \text{ almost everywhere on } \mathcal{S}, \tag{1.1j}$$

where $\lambda = h^2$. As in [27], the object of investigation here will be the system of equations (1.1a)-(1.1j). Since the curve $\mathcal{S}$ is not prescribed a priori, it is called a *free boundary.*

The motivating example for the study of Bernoulli problems occurs in steady hydrodynamic-wave theory, where $\lambda(r) = -2gr$, for some positive constant g. In this context, the free boundary represents the profile of a Stokes wave, i.e. a steady periodic irrotational water wave of infinite depth, with a free surface under gravity and without surface tension, see [33], ψ is a stream function and $(\psi_Y, -\psi_X)$ is a steady velocity field. Then (1.1i) and (1.1j) mean that $\mathcal{S}$ is a streamline at which the pressure in the flow is a constant. In hydrodynamics, a point on $\mathcal{S}$ where the velocity is zero is called a stagnation point.

In his 1847 paper [30], Stokes discussed nonlinear waves with small amplitudes using power series, but he did not show the convergence of these series. In an 1880 note [31], he also conjectured the existence of a large amplitude wave with a stagnation point and a corner containing an angle of 120 degrees at its highest point. He further speculated that this wave of extreme form marks the limit of steady periodic water waves in terms of amplitude (the Stokes wave of greatest height).

The first rigorous mathematical treatment of this free-boundary problem is the local existence theory due to Levi-Civita [13] and Nekrasov [16]. They both used conformal mappings to reduce the question to one of existence of a harmonic function satisfying nonlinear Neumann boundary conditions on a fixed domain. These nonlinear Neumann problems were in turn formulated as nonlinear integral equations.

The first global mathematical treatment is due to Krasovskii [12], who used the Levi-Civita formulation to obtain the existence of wave of all slopes from zero up to, but not including, $\pi/6$. However, he did not show that they form a connected set.

That contribution was due to Keady and Norbury [9], who obtained the existence of a global connected set of Stokes waves. The Nekrasov formulation and global bifurcation theory led to the conclusion that this set contains waves with flow speeds at the wave crest, relative to the wave speed, arbitrarily close to zero. Toland [32] and McLeod [15] showed that in the closure of the continuum found by Keady and Norbury there exists a solution which corresponds to a wave with a stagnation point at its crest. That this wave has indeed a corner at its crest, as Stokes had predicted, was proved independently by Amick, Fraenkel and Toland [1], and by Plotnikov [19].

One can therefore draw the conclusion that in the water-wave problem the free boundary need not be smooth at stagnation points and, in certain situations, it must necessarily have a corner at such a point. On the other hand, Lewy [14] showed that away from stagnation points the profile of a Stokes wave is a real-analytic curve. But many problems concerning Stokes waves with stagnation points remain open: for example it is not known whether a Stokes wave can have uncountably many stagnation points, or even whether there can be more than one stagnation point per (minimal) wavelength. On the positive side, the existence of a Stokes wave with a convex profile between stagnation points has recently been established [21].

For a solution $(\mathcal{S}, \psi)$ of (1.1), a point (X, Y) on $\mathcal{S}$ is called a *stagnation point* if $\lambda(Y) = 0$. We assume throughout that

$$\lambda : (-\infty, 0] \to [0, \infty) \text{ is continuous,} \tag{1.2a}$$

$$\lambda(0) = 0 \qquad \text{and} \qquad \lambda(r) \neq 0 \text{ for all } r \in (-\infty, 0). \tag{1.2b}$$

Nonlinearities λ of this type are a natural generalization of that in the water-wave problem. (However, certain aspects of the theory of Bernoulli free boundaries [27] can be developed for a wider class of nonlinearities than that considered here.) Throughout the thesis the functions λ and h are related by

$$h = \sqrt{\lambda}. \tag{1.3}$$

Since (1.2) holds, it follows that

$$h : (-\infty, 0] \to [0, \infty) \text{ is continuous,} \tag{1.4a}$$

$$h(0) = 0 \qquad \text{and} \qquad h(r) \neq 0 \text{ for all } r \in (-\infty, 0). \tag{1.4b}$$

Under the assumption (1.2), stagnation points, if they exist, are located on the line $Y = 0$, and are necessarily points of maximum height on $\mathcal{S}$.

It is required here that (1.1j) is satisfied in the following sense

$$|\nabla\psi| \text{ is bounded in } \Omega, \tag{1.5a}$$

$$\lim_{(X,Y)\to(u(s),v(s))} |\nabla\psi(X,Y)|^2 = \lambda(v(s)) \quad \text{for almost all } s \in \mathbb{R}, \tag{1.5b}$$

where the limit in (1.5b) is considered as (X, Y) approaches $\mathcal{S}$ from within Ω in non-tangential directions at points where the tangent to the boundary is well-defined (which is the case for almost all s by (1.1b)-(1.1c)). This interpretation of (1.1j) might appear at first to be slightly less general than that in [27], however a careful analysis of the results there shows that, for the type of nonlinearities considered here, the two interpretations are equivalent.

Since it was required that (1.1j) is satisfied in the weak sense (1.5), there arises a question about how smooth free boundaries may be. Let $\mathcal{S}_{\mathcal{N}}$ denote the set of stagnation points. It is proved in [27] that $\mathcal{S}_{\mathcal{N}}$, which is obviously a closed set, has zero measure on $\mathcal{S}$. The local regularity of $\mathcal{S}$ away from stagnation points is now well understood. For example, if λ is real-analytic on $(-\infty, 0)$, it is shown in [27] that $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$ is a union of real-analytic curves, and ψ has a real-analytic extension across $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$, an extension which satisfies (1.1f) pointwise. The case when λ has only a finite degree of differentiability on $(-\infty, 0)$ and its derivatives are locally Hölder continuous was studied in [18], leading to the conclusion that free boundaries are essentially as smooth away from stagnation points as the nonlinearity λ allows.

This thesis can be seen as a further development on [27], in that we investigate the nature of the singularities of free boundaries at stagnation points, a question which was left open there. Of equal interest here is the investigation of geometric properties of free boundaries, whether singular or not. There follows a brief description of the main results obtained.

1.2 The Main Results of the Thesis

This chapter continues with background material, mainly on harmonic analysis in the unit disc. We then describe, following [27], how (1.1) can be studied by means of certain nonlinear pseudo-differential equations for periodic real-valued functions of one real variable. The equations in question involve the conjugation operator $\mathcal{C}$, also known (up to a sign convention) as the periodic Hilbert transform.

In Chapter 2 we first study the local regularity, in Hölder spaces, of solutions to these equations. The results are due to Pichler-Tennenberg [18], but the proof given here is simpler, and independent of [18]. They yield the regularity of $\mathcal{S}\setminus\mathcal{S}_{\mathcal{N}}$, from which the degree of smoothness of ψ in $\Omega \cup (\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}})$ can be deduced.

Then we study geometric properties of free boundaries. We show that, at the highest point of $\mathcal{S}$ one has that $h(Y) < 1$ and, at the lowest point of $\mathcal{S}$, $h(Y) > 1$. This shows that there exists Y_0 with $h(Y_0) = 1$ such that $\mathcal{S}$ intersects the horizontal line $Y = Y_0$. Typically we assume that h is a strictly decreasing function, hence Y_0 with this property is unique. Note that $\mathcal{S}_0 = \{(X, Y_0) : X \in \mathbb{R}\}$ and $\psi_0(X,Y) = Y - Y_0$ is a solution of (1.1), which we call a trivial solution.

Under the assumption that h is decreasing and $\log h$ is concave, we derive some special properties of free boundaries. First, $\mathcal{S}$ must be globally the graph of a function, irrespective of the number of stagnation points on $\mathcal{S}$. This was previously known [27] only when there were at most countably many stagnation points on $\mathcal{S}$ (the result in [27] is different and more general in some other respects). Thus, let $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic. We prove that η is a convex function of X on any interval on which $\eta < Y_0$, where Y_0 is such that $h(Y_0) = 1$. We also prove that, if X_1, X_2 are such that $\eta'(X_1) = \eta'(X_2) = 0$ and $\eta'(X) \neq 0$ for all $X \in (X_1, X_2)$, then

$$|\nabla\psi|(X_1, \eta(X_1)) + |\nabla\psi|(X_2, \eta(X_2)) \leq 2.$$

In particular, $|\nabla\psi| \leq 2$ everywhere on $\mathcal{S}$, and hence everywhere in Ω. We emphasize that this estimate is independent of the log-concave decreasing function h.

We also give some new results in the case when h is decreasing and $\log h$ is convex. If $\mathcal{S}$ is the graph of a function η and X_1, X_2 are such that $\eta'(X_1) =$

$\eta'(X_2) = 0$ and $\eta'(X) \neq 0$ for all $X \in (X_1, X_2)$, then

$$|\nabla\psi|(X_1, \eta(X_1)) + |\nabla\psi|(X_2, \eta(X_2)) \geq 2.$$

Also, η does not have any strict local minimum on any interval on which $\eta > Y_0$, where Y_0 is such that $h(Y_0) = 1$. Another result, of a different nature, provides a condition which ensures that a solution of a certain nonlinear pseudo-differential equation can be used to construct a free boundary.

We then prove a symmetry result. We show that, for any function λ in (1.1j), a free boundary which is globally the graph of a continuous function with exactly one local maximum per period is necessarily symmetric with respect to vertical lines passing through any of its highest points. This result includes the situation when the highest points of the free boundary are stagnation points; in this case, no assumption, apart from continuity, is made on the smoothness of $\mathcal{S}$ and ψ at these points. For smooth solutions of the Stokes-wave problem, such a symmetry result was proved recently by Okamoto and Shoji [17, Theorem 3.6, p. 74], improving on earlier results of Garabedian [7]; see also [34]. However, their method would fail for singular solutions, since there is not enough smoothness of the boundary at stagnation points for their application of the Serrin Corner-Point Lemma. Nevertheless, as in [17], our proof uses the Maximum Principle combined with reflections in vertical lines, in the spirit of the method of moving planes of Alexandrov and Serrin. Other works using this method to prove symmetry of hydrodynamic waves are [3] and [4].

In Chapter 3 we study the behaviour of free boundaries at stagnation points. In the water-wave problem, a profile which is the graph of a continuous even function η, strictly decreasing on $(0, \pi]$ and with $\eta(0) = 0$, corresponds to a 'Stokes wave of extreme form', whose existence was predicted by Stokes [31] and rigorously established by Toland [32] and McLeod [15]. The first Stokes conjecture, which concerns the behaviour of η' at $X = 0$, was elucidated independently by Amick, Fraenkel and Toland [1], and Plotnikov [19], where it was proved that $\lim_{X\to\pm 0} \eta'(X)$ exists and is equal to $\mp 1/\sqrt{3}$. Thus the free boundary has a corner with an included angle of $2\pi/3$. Other examples of singular free boundaries were studied in passing in [21].

One of the main problems that we investigate here is to what extent singular free boundaries must have corners at isolated stagnation points.

We consider first the case when the free boundary is symmetric with respect to the vertical line passing through the stagnation point, assumed to be located at the origin $(0,0)$. We also assume that, in a neighbourhood of the origin, $\mathcal{S}$ is the graph of a continuous function η, which is monotone locally on each side of 0. Under a weak additional assumption on λ, see (1.8) below, we prove the existence of $\lim_{X\to 0^+}\eta'(X)$, and determine its value.

Our approach is based on studying a generalized Nekrasov's equation, see (1.42), satisfied by a 2π-periodic real-valued function θ which, in a suitable parametrization of the free boundary, gives the angle between the tangent to $\mathcal{S}$ and the horizontal. The symmetry of $\mathcal{S}$ implies that θ is an odd function, and the question of interest is the existence of $\lim_{t\to 0^+}\theta(t)$. Suppose that the restriction of h to $(-\delta,0)$ is of class C^1, for some $\delta>0$. Let $f:[0,\infty)\to[0,\infty)$ be given by $f(r)=h(-r)$, for all $r\in[0,\infty)$. Our analysis involves the function $E:(0,\delta)\to\mathbb{R}$ given by

$$E(r):=\frac{f'(r)\int_0^r f(u)\,du}{2\int_0^r f'(u)\,f(u)\,du},\qquad r\in(0,\delta). \tag{1.6}$$

An important family of examples is when $h(r)=c(-r)^\alpha$, for all $r\in(-\infty,0)$, with c and α positive constants. In this case, if θ is odd and satisfies $0\le\theta\le\pi/2$ on $(0,\pi]$, it can be seen by an integration by parts that (1.42) is equivalent to

$$\theta(t)=\mu\int_0^\pi K(t,s)\frac{\sin\theta(s)}{\int_0^s\sin\theta(v)\,dv}\,ds,\qquad t\in(0,\pi], \tag{1.7}$$

where $\mu:=\alpha/(\alpha+1)$, and

$$K(t,s)=\frac{1}{\pi}\log\left|\frac{\sin\frac{1}{2}(t+s)}{\sin\frac{1}{2}(t-s)}\right|,\qquad t,s\in(0,\pi].$$

In particular, in the water-wave problem $\mu=1/3$ and (1.7) is Nekrasov's integral equation, for which it was proved in [1] that any solution with $0\le\theta\le\pi/2$ on $(0,\pi]$ satisfies $\lim_{t\to 0^+}\theta(t)=\pi/6$. Under a less restrictive assumption on θ, a similar result was proved, by completely different methods, in [19]. More recently, equation (1.7) with $\mu\in[1/3,1)$ was studied by Plotnikov and Toland [21], where they proved the second Stokes conjecture, on the existence of a convex Stokes wave of extreme form. As part of that proof, it was shown that any solution of (1.7) with $0 \leq \theta \leq \pi/2$ on $(0, \pi]$ satisfies $\lim_{t \to 0^+} \theta(t) = \mu\pi/2$, using the same method as in [1]. Notice that, in this family of examples, the function E in (1.6) is identically equal to μ.

Here we give a substantial generalization of these results, in which we require only that θ is odd and there exists $t_0 \in (0, \pi]$ such that θ is continuous on $(0, t_0)$ and $0 \leq \theta \leq \pi/2$ on $(0, t_0)$, conditions more general than those implied by the symmetry of $\mathcal{S}$ and the local monotonicity of η. The key assumption is that the function E defined in (1.6) satisfies

$$\text{there exists } \mu \in (0,1) \text{ such that } \lim_{r \to 0^+} E(r) = \mu. \tag{1.8}$$

The main result is that, if (1.8) holds, then any such solution of (1.42) satisfies $\lim_{t \to 0^+} \theta(t) = \mu\pi/2$, which means that, for the corresponding free boundary, $\lim_{X \to 0^+} \eta'(X) = -\tan(\mu\pi/2)$. Our assumptions admit the possibility of other stagnation points, even uncountably many, on the free boundary. For periodic water waves, our result has the same generality as [19]. However, the proof in [19] is both difficult and specific to the special case of water waves, and does not extend to other situations, for example for non-real-analytic nonlinearities λ in (1.1j). The proof here follows the lines of [1] and [21], but with some significant simplification. It is shown first that the existence of $\lim_{t \to 0^+} \theta(t)$ can be reduced to a uniqueness question for an integral equation on $(0, \infty)$. This equation does not depend on the function λ, but only on the value of μ associated to it in (1.8). The proof of the required uniqueness then simplifies and extends the proofs in [1] and [21].

We then construct an example, inspired by [1, Appendix], of a symmetric singular Bernoulli free boundary $\mathcal{S}$, for a function λ with reasonable smoothness properties, such that $\mathcal{S}$ does not have a corner at the singular point. The only possible explanation for this happening is that (1.8) fails, although we have not been able to check this directly.

To end Chapter 3, we investigate the shape of free boundaries close to isolated asymmetric stagnation points, if such exist. We show that, if there exist lateral tangents at such a point, then these are symmetric with respect to the vertical line passing through that point, and the angle enclosed has the same size as for a symmetric stagnation point corresponding to the same nonlinearity h. It remains an open question whether there exist asymmetric stagnation points where the free boundary does not have a corner, but instead has a more complicated shape. To describe the geometry of Bernoulli free boundaries with infinitely many singular points on a period, if such exist, would be a more formidable task.

In Chapter 4 we restrict attention to the water-wave problem and show, using modern methods, how the existence of a singular solution of Nekrasov's equation can be deduced from the existence of a suitable sequence of smooth solutions, under more general assumptions than those used by Toland [32] and McLeod [15]. We also show that a Gibbs Phenomenon, originally exhibited in [15], which occurs as smooth solutions approach a singular solution, is still displayed in the present wider setting. We also prove a new result on the asymptotic behaviour of solutions of McLeod's 'boundary layer equation' [15].

Chapter 5 is devoted to extending the argument of Plotnikov and Toland [20] on the existence of an extension to a whole strip of a harmonic function originally defined on a half-strip, using complex ODE. Whilst this result requires strong assumptions on the nonlinearity h, it is quite important, since it has proved to be one of the essential ingredients in the recent proof of the convexity of the Stokes wave of extreme form [21]. A consequence of this approach is that the Fourier coefficients of θ form a log-convex sequence, where θ is the angle between the tangent to $\mathcal{S}$ and the horizontal.

The results of Section 2.1, Section 2.3 and Section 3.2 have been included in the paper [37] which has been submitted for publication.

1.3 Background Material

We start by collecting notation and recalling some notions and classical results, mainly from harmonic analysis in the unit disc, see [5] and [10].

We denote by $L^p_{2\pi}$, $0 < p < \infty$, the space of 2π-periodic locally p^{th}-power summable real-valued 'functions', and by $L^\infty_{2\pi}$ the space of 2π-periodic essentially bounded real-valued 'functions'. For $1 \leq p \leq \infty$, $L^p_{2\pi}$ is a Banach space when endowed with the norm

$$||u||_{L^p_{2\pi}} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(t)|^p\, dt\right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty,$$

and

$$||u||_{L^\infty_{2\pi}} = \operatorname{ess\,sup}\{|u(t)| : t \in \mathbb{R}\}.$$

For $u \in L^1_{2\pi}$, let

$$[u] = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\,dt. \tag{1.9}$$

For $1 \le p \le \infty$, let $W^{1,p}_{2\pi}$ be the Sobolev space of absolutely continuous 2π-periodic functions u with weak first derivatives $u' \in L^p_{2\pi}$.

Let D denote the open unit disc centred at 0 in the complex plane. For any function $W : D \to \mathbb{C}$, let

$$W_r(t) = W(re^{it}) \quad \text{for } t \in \mathbb{R} \text{ and } r \in (0,1).$$

Let also $W^*(t) := \lim_{r \nearrow 1} W(re^{it})$, whenever this limit exists.

For any $z \in D$, $z = re^{it}$, let us consider the Poisson kernel

$$P_r(t) = \operatorname{Re}\frac{1+z}{1-z} = \frac{1-r^2}{1+r^2-2r\cos t}. \tag{1.10}$$

Then, for any $u \in L^1_{2\pi}$, the function $U : D \to \mathbb{C}$ given by Poisson formula

$$U(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-s)u(s)\,ds, \tag{1.11}$$

is harmonic in D, U^* is well-defined a.e., and $U^* = u$ a.e..

A harmonic conjugate V of U in D, normalized so that $V(0) = 0$, is obtained by the formula

$$V(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q_r(t-s)u(s)\,ds, \tag{1.12}$$

where $Q_r(t)$ is the conjugate Poisson kernel

$$Q_r(t) = \operatorname{Im}\frac{1+z}{1-z} = \frac{2r\sin t}{1+r^2-2r\cos t}. \tag{1.13}$$

Then V^* exists a.e., and the *conjugate function* Cu is defined a.e. by $Cu := V^*$.

It is a basic result of harmonic analysis that, for any $u \in L^1_{2\pi}$, the conjugate function Cu can be calculated almost everywhere as a Cauchy Principal Value

Integral

$$\mathcal{C}u(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\Big(\frac{1}{2}(x-y)\Big)u(y)\,dy. \tag{1.14}$$

As an operator acting on $L^1_{2\pi}$, $\mathcal{C}$ does not map $L^1_{2\pi}$ into itself but, if $u \in L^1_{2\pi}$, then $\mathcal{C}u \in L^p_{2\pi}$ for all $p \in (0,1)$. However, we have

Theorem 1.1 (M. Riesz, [10, Chapter V, §B]**).** *For any $p \in (1,\infty)$, $\mathcal{C}$ is a bounded linear operator from $L^p_{2\pi}$ into itself.*

The *real Hardy space* $\mathcal{H}^1_{\mathbb{R}}$ is the set of functions $u \in L^1_{2\pi}$ with $\mathcal{C}u \in L^1_{2\pi}$. It is a Banach space with the norm $||u||_{\mathcal{H}^1_{\mathbb{R}}} = ||u||_1 + ||\mathcal{C}u||_1$.

We consider also the space $\mathcal{H}^{1,1}_{\mathbb{R}} := \{u \in W^{1,1}_{2\pi} : u' \in \mathcal{H}^1_{\mathbb{R}}\}$, which is a Banach space with the norm $||u||_{\mathcal{H}^{1,1}_{\mathbb{R}}} = ||u||_{\mathcal{H}^1_{\mathbb{R}}} + ||u'||_{\mathcal{H}^1_{\mathbb{R}}}$.

For any $u \in \mathcal{H}^1_{\mathbb{R}}$, $\mathcal{C}(\mathcal{C}u) = -u + [u]$.

For $0 < p \leq \infty$, the *Hardy class* $\mathcal{H}^p_{\mathbb{C}}$ is the set of holomorphic functions $W : D \to \mathbb{C}$ such that

$$\sup_{r\in(0,1)} ||W_r||_{L^p_{2\pi}} < \infty.$$

If $W \in \mathcal{H}^p_{\mathbb{C}}$, $0 < p \leq \infty$, then $W^*(t) := \lim_{r\nearrow 1} W(re^{it})$ is well defined almost everywhere, satisfies $W^* \in L^p_{2\pi}$, and $\log|W^*| \in L^1_{2\pi}$ if $W \not\equiv 0$.

If $W \in \mathcal{H}^1_{\mathbb{C}}$ and $u = \operatorname{Im} W^*$, then $u \in \mathcal{H}^1_{\mathbb{R}}$ and $W^* = -(\gamma + \mathcal{C}u) + iu$, where $\gamma := -\operatorname{Re} W(0)$. Conversely, if $u \in \mathcal{H}^1_{\mathbb{R}}$ and $\gamma \in \mathbb{R}$, there exists $W \in \mathcal{H}^1_{\mathbb{C}}$ such that $W^* = -(\gamma + \mathcal{C}u) + iu$.

An *outer function* G is a holomorphic function on D which can be written in the form $G = \alpha\,\mathcal{O}(g)$ where $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and

$$\mathcal{O}(g)(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log|g|(t)dt\right\}, \quad z \in D, \tag{1.15}$$

for some 2π-periodic function $g : \mathbb{R} \to \mathbb{C}$ with $\log|g| \in L^1_{2\pi}$.

Outer functions have the following properties [23, Theorems 17.16 & 17.7]:

$|(\mathcal{O}(g))^*| = |g|$ for any g with $\log|g| \in L^1_{2\pi}$, and, for $p \in (0,\infty]$,
$|g| \in L^p_{2\pi}$ if and only if $\mathcal{O}(g) \in \mathcal{H}^p_{\mathbb{C}}$;

for $W \in \mathcal{H}^p_{\mathbb{C}}$, $p \in (0,\infty]$ and $z \in D$, $|\mathcal{O}(W^*)(z)| \geq |W(z)|$;

for $W \in \mathcal{H}^p_{\mathbb{C}}$, $p > 0$, W is an outer function if and only if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|W^*|\,ds = \log|W(0)|;$$

if W and $1/W$ are in H^p, $p > 0$, then W is an outer function.

Let Π be a subset of $\mathbb{R}^m$, $m \in \{1,2\}$. For any $\alpha \in (0,1)$, the Hölder space $C^{0,\alpha}(\Pi)$ is the set of functions $v : \Pi \to \mathbb{R}$ which satisfy

$$||v||_{C^{0,\alpha}(\Pi)} := \sup_{x\in\Pi} |v(x)| + \sup_{x_1,x_2\in\Pi, x_1\neq x_2} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^\alpha} < \infty.$$

Such functions v are called Hölder continuous with exponent α in Π.

Let Ξ be an open set in $\mathbb{R}^m$, $m \in \{1,2\}$. For $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1)$, the space $C^{n,\alpha}(\Xi)$ is the set of functions $v : \Xi \to \mathbb{R}$ which are n times continuously differentiable in Ξ, and v and all its partial derivatives up to order n are in $C^{0,\alpha}(\Xi)$. The space $C^{n,\alpha}_{\rm loc}(\Xi)$ is the set of functions $v : \Xi \to \mathbb{R}$ such that v and its derivatives are Hölder continuous with exponent α on any compact subset of Ξ.

Let Θ be an open set in $\mathbb{R}^2$, and Υ be a subset of the boundary of Θ which is open in the relative topology of $\partial\Theta$. For $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1)$, let $C^{n,\alpha}_{\rm loc}(\Theta \cup \Upsilon)$ denote the space of functions v in $C^{n,\alpha}_{\rm loc}(\Theta)$ with the additional property that v and all its partial derivatives up to order n have continuous extensions to $M := \Theta \cup \Upsilon$, and every point in Υ has a neighbourhood (relative to M) in which v and its derivatives are Hölder continuous with exponent α. (We say that a holomorphic function $W : \Theta \to \mathbb{C}$ is in $C^{n,\alpha}_{\rm loc}(\Theta \cup \Upsilon)$ if its real and imaginary part are in $C^{n,\alpha}_{\rm loc}(\Theta \cup \Upsilon)$.)

Let $C^{n,\alpha}_{2\pi}$ be the space of 2π-periodic functions in $C^{n,\alpha}_{\rm loc}(\mathbb{R})$.

Theorem 1.2 (Privalov, [10, Chapter V, §E]**).** *$\mathcal{C}$ is a bounded linear operator from $C^{0,\alpha}_{2\pi}$ into itself, for all $\alpha \in (0,1)$.*

The following result is a local version of Privalov's Theorem. A hint for the proof is given in [27, Remark A.2, Appendix].

Lemma 1.3. *Let $\alpha \in (0,1)$, and suppose that $[c,d] \subset (a,b)$, where $a,b,c,d \in \mathbb{R}$. Then $\mathcal{C}u|_{[c,d]} \in C^{0,\alpha}([c,d])$ for any $u \in L^1_{2\pi}$ with $u|_{[a,b]} \in C^{0,\alpha}([a,b])$, and there* *exists a constant K, independent of u, such that*

$$||\mathcal{C}u||_{C^{0,\alpha}([c,d])} \leq K\big(||u||_{L^1_{2\pi}} + ||u||_{C^{0,\alpha}([a,b])}\big).$$

If I is any open interval of the real line, let $\Gamma_I := \{e^{it} : t \in I\}$ be the corresponding open arc on the unit circle.

Lemma 1.4. *Let $I \subset \mathbb{R}$ be an open interval, and $u \in \mathcal{H}^1_{\mathbb{R}} \cap C^{k,\alpha}_{\text{loc}}(I)$, where $k \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < 1$. Then $\mathcal{C}u \in C^{k,\alpha}_{\text{loc}}(I)$. Furthermore, if the holomorphic function $W \in \mathcal{H}^1_{\mathbb{C}}$ is such that $W^* = -\mathcal{C}u + iu$, then $W \in C^{k,\alpha}_{\text{loc}}(D \cup \Gamma_I)$.*

Proof of Lemma 1.4. If the length of I is greater than 2π, then u is actually in $C^{k,\alpha}_{2\pi}$, and the result follows by induction on k. Indeed, for $k = 0$ the result is immediate from Privalov's Theorem and the fact that the Poisson integral of a $C^{0,\alpha}_{2\pi}$ function is in $C^{0,\alpha}(\overline{D})$, see [6, Theorem B.30, p. 260]; alternatively, see [22, Proposition 3.4, p. 47]. For $k \geq 1$, one uses the fact that $\mathcal{C}$ commutes with differentiation on $C^{1,\alpha}_{2\pi}$, see [35, Appendix], and also the result in [5, Theorem 3.11, p. 42] which shows the connection between the boundary values of a holomorphic function in D and those of its complex derivative.

Suppose now that the length of I does not exceed 2π, and let J be any open interval whose closure is contained in I. Then one can write $u = u_1 + u_2$, where $u_1 \in C^{k,\alpha}_{2\pi}$ and $u_2 \equiv 0$ on J. Clearly $u_1 \in \mathcal{H}^1_{\mathbb{R}}$ and hence $u_2 \in \mathcal{H}^1_{\mathbb{R}}$ also. Let $W_1, W_2 \in \mathcal{H}^1_{\mathbb{C}}$ be such that $W_j^* = -\mathcal{C}u_j + iu_j$, $j = 1, 2$, so that $W = W_1 + W_2$. Then $\mathcal{C}u_1 \in C^{k,\alpha}_{2\pi}$, and $W_1 \in C^{k,\alpha}(\overline{D})$. Also, the fact that $u_2 \equiv 0$ on J implies by [10, Chapter III, §E] that W_2 has an holomorphic extension to $D \cup \Gamma_J \cup (\mathbb{C} \setminus \overline{D})$ and, as a consequence, $\mathcal{C}u_2$ is real-analytic on J. Since this holds for all $J \subset I$, the result follows. □

Lemma 1.5. *Let $u \in \mathcal{H}^1_{\mathbb{R}}$ and suppose that there exists $\delta \in (0, \pi)$ and $\varepsilon > 0$ such that*

$$u(s) - u(-s) \geq \varepsilon \quad \textit{for all} \quad s \in (0, \delta). \tag{1.16}$$

Then $\mathcal{C}u$ is unbounded on any interval $(-\gamma, \gamma)$, $\gamma > 0$. The same conclusion holds if (1.16) is replaced by

$$u(s) - u(-s) \leq -\varepsilon \quad \textit{for all} \quad s \in (0, \delta). \tag{1.17}$$

Proof of Lemma 1.5. Let U be the Poisson integral of u, and V be a harmonic conjugate of U, normalized so that $V(0) = 0$. Then (1.12) applied with $t = 0$ yields

$$V(re^{i0}) = \frac{1}{\pi}\int_0^\delta \frac{(u(-s)-u(s))r\sin s}{1-2r\cos s+r^2}\,ds + \frac{1}{\pi}\int_\delta^\pi \frac{(u(-s)-u(s))r\sin s}{1-2r\cos s+r^2}\,ds. \quad (1.18)$$

The second integral in (1.18) tends to a finite limit as $r \nearrow 1$. Since

$$\int_0^\delta \frac{r\sin s}{1-2r\cos s+r^2}\,ds = \operatorname{Im}\left(\int_0^\delta \frac{-e^{is}}{e^{is}-r}\,ds\right) = \log\left|\frac{e^{i\delta}-r}{1-r}\right|, \quad (1.19)$$

it follows that, as $r \nearrow 1$, the first integral in (1.18) tends to $-\infty$ if (1.16) holds, and it tends to $+\infty$ if (1.17) holds. In either case, we conclude that $V(re^{i0})$ is unbounded as $r \nearrow 1$.

On the other hand, V is the Poisson integral of Cu. If Cu were bounded on some interval $(-\gamma, \gamma)$, it would be immediate from Poisson formula (1.11) that $V(re^{i0})$ is bounded as $r \nearrow 1$, a contradiction. This concludes the proof of the Lemma. □

The following result is a slightly more general version of [27, Lemma 3.2], with the same proof.

Lemma 1.6. *Let $\mathcal{T} = \{(u(t), v(t)) : t \in \mathbb{R}\}$ be a non-self-intersecting continuous curve such that*

$$u(t)^2 + v(t)^2 \to \infty \quad \text{as} \quad t \to \pm\infty,$$

and let Σ be one of the two connected components of $\mathbb{C} \setminus \mathcal{T}$. Suppose $\tilde{\psi} \not\equiv 0$ is harmonic and bounded above in Σ, continuous in $\Sigma \cup \mathcal{T}$ and $\tilde{\psi} \equiv 0$ on $\mathcal{T}$. Let $\tilde{\varphi}$ be a harmonic conjugate of $-\tilde{\psi}$. Then $\tilde{\varphi} + i\tilde{\psi}$ is a conformal bijection from Σ onto the lower half-plane $\mathbb{C}_-$ which maps ∞ onto ∞. Moreover $\tilde{\phi} + i\tilde{\psi}$ can be extended as a homeomorphism from the closure $\Sigma \cup \mathcal{T}$ of Σ onto the closure $\mathbb{R} \cup \mathbb{C}_-$ of $\mathbb{C}_-$.

We now recall a version of the notion of equicontinuity for a family of continuous function on a metric space, and the classical Ascoli-Arzela Theorem.

Let (X, d) be a metric space, and let $C(X)$ be the space of real-valued continuous function on X. Let $\mathcal{F}$ be a subset of $C(X)$. We say that that $\mathcal{F}$ is (or its elements are) *equicontinuous* at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever $x \in X$ and $d(x, x_0) < \delta$, then

$$|u(x) - u(x_0)| < \varepsilon \quad \text{for all} \;\; u \in \mathcal{F}.$$

We say that $\mathcal{F}$ is equicontinous on X if it is equicontinuous at every point of X.

When (X, d) is a compact metric space, $C(X)$ endowed with the supremum norm is a Banach space.

Theorem 1.7 (Ascoli-Arzela). *Let (X, d) be a compact metric space, and let $\mathcal{F}$ be a subset of $C(X)$. Then $\mathcal{F}$ is relatively compact in $C(X)$ if and only if $\mathcal{F}$ is uniformly bounded and equicontinuous on X.*

1.4 Basic Theory of Bernoulli Free Boundaries

We now recall how the study of the free-boundary problem (1.1) is equivalent to the study of certain nonlinear pseudo-differential equations for periodic functions of one real variable. We just sketch the derivation of these equations which will be used throughout the thesis. More details and full proofs are to be found in [27]. However the presentation here is specifically designed so as to suit the purposes of the later work in the thesis, and in some respects is different from that in [27].

Let $(\mathcal{S}, \psi)$ be a solution (1.1a)-(1.1j). Let φ be a harmonic conjugate of $-\psi$ in Ω, so that $\omega := \varphi + i\psi$ is a holomorphic function, where the (X, Y)-plane is identified with the complex plane of generic variable $Z = X + iY$. Then ω is an injective conformal mapping of Ω onto the open lower half-plane $\mathbb{C}_- := \{z = x + iy : x \in \mathbb{R},\, y < 0\}$, which maps ∞ onto ∞, and can be extended as a homeomorphism of $\Omega \cup \mathcal{S}$ onto the closed lower half-plane. Moreover, $\omega(X + iY) - (X + iY)$ is 2π-periodic in X.

Let Z be the inverse conformal mapping, a homeomorphism of the closed lower half-plane onto $\overline{\Omega}$, with $Z(z) - z$ a 2π-periodic function of z. It follows from Morera's Theorem that the holomorphic function $V : D \setminus [-1, 0] \to \mathbb{C}$ defined by

$$V(\zeta) = Z(i \log \zeta) - i \log \zeta \tag{1.20}$$

has a holomorphic extension to $D \setminus \{0\}$. (Note that the mapping $\zeta \mapsto i \log \zeta$ is a conformal bijection from $D \setminus [-1, 0]$ to the half-strip $\{z = x + iy : -\pi < x <$

π, $y < 0\}$.) The condition (1.1h) can be used to show that the singularity of V at 0 is removable, so that V has an extension as an analytic function in D. Furthermore, V extends continuously to $\overline{D}$. Let

$$w(t) := \operatorname{Im} V^*(t) \qquad \text{for all } t \in \mathbb{R}, \tag{1.21}$$

where $V^*(t) = \lim_{r \nearrow 1} V(re^{it})$. Then

$$-(\gamma + \mathcal{C}w(t)) + iw(t) = V^*(t) = Z(-t) + t \qquad \text{for all } t \in \mathbb{R}, \tag{1.22}$$

where $\gamma \in \mathbb{R}$ is a constant. It is clear that $\mathcal{S} = \{Z(-t) : t \in \mathbb{R}\}$, which means that

$$\mathcal{S} = \{(-(\gamma + t + \mathcal{C}w(t)), w(t)) : t \in \mathbb{R}\}, \tag{1.23}$$

and hence

$$t \mapsto (-(t + \mathcal{C}w(t)), w(t)) \quad \text{is injective on } \mathbb{R}. \tag{1.24}$$

The local rectifiability of $\mathcal{S}$ yields that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ and, if $W \in \mathcal{H}^1_{\mathbb{C}}$ is such that

$$W^* = w' + i(1 + \mathcal{C}w'), \tag{1.25a}$$

then, by (1.5),

$$1/W \in \mathcal{H}^\infty_{\mathbb{C}} \tag{1.25b}$$

$$\lambda(w)\{w'^2 + (1 + \mathcal{C}w')^2\} = 1 \quad \text{almost everywhere,} \tag{1.25c}$$

where $h(w) \neq 0$ almost everywhere and $1/h(w) \in L^1_{2\pi}$. Let

$$\mathcal{N} := \{t \in \mathbb{R} : h(w(t)) = 0\} = \{t \in \mathbb{R} : w(t) = 0\}. \tag{1.26}$$

Then the set of stagnation points on $\mathcal{S}$ is given by

$$\mathcal{S}_{\mathcal{N}} = \{(-(\gamma + t + \mathcal{C}w(t)), w(t)) : t \in \mathcal{N}\}.$$

In what follows, the elements of $\mathcal{N}$ will also be referred to as stagnation points. As we have seen, $\mathcal{N}$ has zero measure and, since h and w are continuous functions, $\mathcal{N}$ is a closed set.

We have therefore just explained how solutions $(\mathcal{S}, \psi)$ of (1.1) give rise to solutions $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of (1.25), and such that (1.24) holds.

Conversely, suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (1.25) and that (1.24) holds. Let $\mathcal{S}$ be defined by (1.23), where $\gamma \in \mathbb{R}$, and Ω be the domain below $\mathcal{S}$. Then there exists a conformal mapping ω of Ω onto $\mathbb{C}_-$ such that, if $\psi = \operatorname{Im} \omega$, then $(\mathcal{S}, \psi)$ is a solution of (1.1).

Let us mention in passing that an equivalent form of (1.25) is, see [27],

$$\lambda(w)(1 + \mathcal{C}w') + \mathcal{C}(\lambda(w)w') = 1 \quad \text{almost everywhere,} \tag{1.27}$$

which generalizes Babenko's equation in the theory of water waves. However, in this thesis we shall make no use of the equation in this form.

We now explain how the study of (1.25) is equivalent to that of a system of equations which provides a natural generalization of Nekrasov's equation from water-wave theory.

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfy (1.25). Since $W \in \mathcal{H}^1_{\mathbb{C}}$ and $1/W \in \mathcal{H}^\infty_{\mathbb{C}}$, it follows that W is an outer function, and

$$W = i\mathcal{O}(1/h(w)), \tag{1.28}$$

where $1/h(w) \in L^1_{2\pi}$ and $h(w) \in L^\infty_{2\pi}$. Let τ, ϑ be defined by

$$\tau := -\log h(w), \tag{1.29a}$$

$$\vartheta := -\mathcal{C}\tau. \tag{1.29b}$$

Then $\tau \in L^p_{2\pi}$ for all $p \in (1, \infty)$, and M. Riesz' Theorem shows that $\vartheta \in L^p_{2\pi}$ for all $p \in (1, \infty)$. Since $W(0) = i$, it follows that

$$[\tau] = 0. \tag{1.29c}$$

We deduce from (1.28) that

$$W^* = \frac{i \exp\{-i\mathcal{C}(\log h(w))\}}{h(w)} = \frac{\sin \vartheta}{h(w)} + i \frac{\cos \vartheta}{h(w)},$$

from where we conclude using (1.25a) that, almost everywhere,

$$h(w)w' = \sin\vartheta, \tag{1.30a}$$

$$h(w)(1+\mathcal{C}w') = \cos\vartheta. \tag{1.30b}$$

Observe now from (1.30) that, if $\mathcal{S}$ represents a free boundary parametrized by (1.23) then, for almost all $t \in \mathbb{R}$, $-\vartheta(t)$ represents the angle which the tangent to $\mathcal{S}$ at the point $(-(\gamma+t+\mathcal{C}w(t)), w(t))$ makes with the horizontal.

Let $H : (-\infty, 0] \to (-\infty, 0]$ be given by

$$H(r) = -\int_r^0 h(u)\,du, \quad r \in (-\infty, 0]. \tag{1.31}$$

It follows from integrating (1.30a) that

$$H(w(t)) - H(w(0)) = \int_0^t \sin\vartheta(v)\,dv \qquad \text{for all } t \in \mathbb{R}.$$

Since H is invertible, it follows that

$$w(t) = H^{-1}\Big(-\nu + \int_0^t \sin\vartheta(v)\,dv\Big) \qquad \text{for all } t \in \mathbb{R}, \tag{1.32}$$

where $\nu := -H(w(0))$, $\nu \geq 0$. Clearly $-\nu + \int_0^t \sin\vartheta(v)\,dv \leq 0$ for all $t \in \mathbb{R}$, and $-\nu + \int_0^t \sin\vartheta(v)\,dv = 0$ if and only if $t \in \mathcal{N}$.

Hence, from (1.32) and (1.29) we get that τ, ϑ satisfy the system of equations

$$\vartheta = -\mathcal{C}\tau, \tag{1.33a}$$

$$\tau(s) = -\log\Big\{h\Big(H^{-1}\Big(-\nu + \int_0^s \sin\vartheta(v)\,dv\Big)\Big)\Big\}, \quad s \in \mathbb{R}, \tag{1.33b}$$

$$[\tau] = 0. \tag{1.33c}$$

and the integrability condition

$$\Big\{s \mapsto \frac{1}{h(H^{-1}(-\nu + \int_0^s \sin\vartheta(v)\,dv))}\Big\} \in L^1_{2\pi}, \tag{1.33d}$$

where, in (1.33b), $\nu \geq 0$ and $-\nu \in \mathcal{R}(H)$.

Conversely, let τ, ϑ satisfy (1.33), where h and H are related by (1.31). Now let $u \in W^{1,1}_{2\pi}$ be given by

$$u(s) = H^{-1}\Big(-\nu + \int_0^s \sin\vartheta(v)\, dv \Big) \quad \text{for all } s \in \mathbb{R}, \tag{1.34}$$

so that $1/h(u) \in L^1_{2\pi}$, and let an outer function W be given by

$$W = i\mathcal{O}(1/h(u)). \tag{1.35}$$

Moreover, $W \in \mathcal{H}^1_{\mathbb{C}}$ and $1/W \in \mathcal{H}^\infty_{\mathbb{C}}$. Since $W \in \mathcal{H}^1_{\mathbb{C}}$ and $W(0) = i$, there exists $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$, w determined up to an additive constant, such that

$$W^* = w' + i(1 + \mathcal{C}w'). \tag{1.36}$$

It follows from (1.35) and (1.36) that

$$w' + i(1 + \mathcal{C}w') = \frac{\sin\vartheta}{h(u)} + i\,\frac{\cos\vartheta}{h(u)}, \tag{1.37}$$

so that

$$h(u)w' = \sin\vartheta \quad \text{almost everywhere.}$$

Now differentiating the formula for u gives

$$h(u)u' = \sin\vartheta \quad \text{almost everywhere.}$$

Hence $u' = w'$ almost everywhere, and we may now fix w by putting $w = u$. We get from (1.37) that

$$w' + i(1 + \mathcal{C}w') = \frac{\sin\vartheta}{h(w)} + i\,\frac{\cos\vartheta}{h(w)}, \tag{1.38}$$

from where it is obvious that (1.25c) holds. We have therefore shown how solutions of (1.33) give rise to solutions of (1.25).

It is sometimes more convenient in the system (1.33) to work with

$$\theta := -\vartheta. \tag{1.39}$$

instead of ϑ. To this end, let $f,\ F : [0,\infty) \to [0,\infty)$ be defined, for all $r \in [0,\infty)$, by

$$f(r) := h(-r), \tag{1.40}$$

and

$$F(r) := \int_0^r f(u)\,du, \tag{1.41}$$

so that $F(r) = -H(-r)$.

It is immediate from (1.33) that

$$\theta = \mathcal{C}\tau, \tag{1.42a}$$

$$\tau(s) = -\log\left\{f\left(F^{-1}\Big(\nu + \int_0^s \sin\theta(v)\,dv\Big)\right)\right\}, \quad s \in \mathbb{R}, \tag{1.42b}$$

$$[\tau] = 0, \tag{1.42c}$$

where $\nu \geq 0$, $\nu \in \mathcal{R}(F)$, and

$$\left\{s \mapsto \frac{1}{f(F^{-1}(\nu + \int_0^s \sin\theta(v)\,dv))}\right\} \in L^1_{2\pi}. \tag{1.42d}$$

Obviously the above derivation of (1.42) from (1.33) can be reversed, in the sense that, if $(\tau,\ \theta)$ is a solution of (1.42) for some given $f,\ F : [0,\infty) \to [0,\infty)$ related by (1.41), then, with ϑ given by (1.39), $(\tau,\ \vartheta)$ is a solution of (1.33), for $h : (-\infty, 0] \to [0,\infty)$ given by (1.40).

Chapter 2

Regularity and Geometric Properties of Free Boundaries

This chapter starts with a straightforward derivation of a regularity theory, in Hölder spaces, of free boundaries away from stagnation points. We then study geometric properties of free boundaries. When h is decreasing and log-concave, we show that any free boundary must necessarily be globally the graph of a function, even if there are uncountably many stagnation points, and that the function in question is convex on any interval on which its graph lies below a certain line. For the same class of problems, we exhibit the bound $|\nabla\psi| \leq 2$ in Ω, which is derived from a result relating the values of $|\nabla\psi|$ at consecutive local extrema of the height of the free boundary. Results of a related type are shown to hold when h is decreasing and log-convex. We conclude the chapter with a result showing that, even in the presence of singularities, a geometrically simple Bernoulli free boundary is necessarily symmetric.

2.1 Local Regularity

We now present a local regularity theory for solutions of (1.29)-(1.30) in spaces of functions with finitely many derivatives which are locally Hölder continuous. Then we seek to infer from this, in certain situations, the smoothness of $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$ and the regularity of ψ in $\Omega \cup (\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}})$. Whilst many of the arguments in the rest of this section are reasonably standard, they are included for completeness.

The following result is similar in some respects to [18, Theorem 2.1].

Theorem 2.1. *Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ and τ, $\vartheta \in L^p_{2\pi}$ for all $p \in (1,\infty)$ satisfy (1.29)-(1.30). Let $\hat{I} \subset (-\infty, 0)$ be an open interval such that $h \in C^{n,\alpha}_{\text{loc}}(\hat{I})$, where $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1)$, and let $I \subset \mathbb{R}$ be an open interval such that $w(I) \subset \hat{I}$. Then $w \in C^{n+1,\alpha}_{\text{loc}}(I)$ and τ, $\vartheta \in C^{n,\alpha}_{\text{loc}}(I)$.*

Proof of Theorem 2.1. We use a simple bootstrap argument. It follows from (1.30a) that $w \in W^{1,\infty}_{\text{loc}}(I)$, and hence $\log h(w) \in C^{0,\alpha}_{\text{loc}}(I)$, so that $\tau \in C^{0,\alpha}_{\text{loc}}(I)$. By the first part of Lemma 1.4, $\vartheta \in C^{0,\alpha}_{\text{loc}}(I)$ and hence, by (1.30a), $w \in C^{1,\alpha}_{\text{loc}}(I)$. Suppose now that τ, $\vartheta \in C^{k-1,\alpha}_{\text{loc}}(I)$ and $w \in C^{k,\alpha}_{\text{loc}}(I)$ for some k with $1 \leq k \leq n$. Since $h \in C^{n,\alpha}_{\text{loc}}(\hat{I})$ by hypothesis, it follows that $\log h(w) \in C^{k,\alpha}_{\text{loc}}(I)$, so that $\tau \in C^{k,\alpha}_{\text{loc}}(I)$. By the first part of Lemma 1.4, $\vartheta \in C^{k,\alpha}_{\text{loc}}(I)$ and therefore, by (1.30a), $w \in C^{k+1,\alpha}_{\text{loc}}(I)$. The conclusion of Theorem 2.1 is immediate.

□

Consider now a solution $(\mathcal{S}, \psi)$ of (1.1), where $h \in C^{n,\alpha}_{\text{loc}}(-\infty, 0)$ for some $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1)$. This gives rise to a solution of (1.29)-(1.30). Theorem 2.1 and Lemma 1.4 show that w, $\mathcal{C}w \in C^{n+1,\alpha}_{\text{loc}}(I)$ for all open intervals I contained in $\mathbb{R} \setminus \mathcal{N}$. Moreover, by (1.22) and Lemma 1.4, the holomorphic function V is in $C^{n+1,\alpha}_{\text{loc}}(D \cup \Gamma_I)$ for all such intervals I. Since w' and $\mathcal{C}w'$ are continuous away from stagnation points, it follows that (1.25c) must hold everywhere on $\mathbb{R} \setminus \mathcal{N}$, and in particular

$$w'(t)^2 + (1 + \mathcal{C}w'(t))^2 \neq 0 \quad \text{for all } t \in \mathbb{R} \setminus \mathcal{N}. \tag{2.1}$$

It is clear from this and (1.23) that, in a neighbourhood of any point of $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$, $\mathcal{S}$ is a curve of class $C^{n+1,\alpha}$; for brevity we say that $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$ is a $C^{n+1,\alpha}_{\text{loc}}$ curve.

Note also that, if $\widetilde{\mathcal{N}} := \{-t : t \in \mathcal{N}\}$, then (1.20) shows that the holomorphic function Z is in $C^{n+1,\alpha}_{\text{loc}}(\mathbb{C}_- \cup (\mathbb{R} \setminus \widetilde{\mathcal{N}}))$. Let X and Y denote the real and imaginary parts of Z, so that $X, Y \in C^{n+1,\alpha}_{\text{loc}}(\mathbb{R}^2_- \cup (\mathbb{R} \setminus \widetilde{\mathcal{N}}))$. A standard higher-order reflection, see e.g. [8, Lemma 6.37, p. 136], yields extensions of X, Y as functions $\hat{X}, \hat{Y} \in C^{n+1,\alpha}_{\text{loc}}(\mathbb{R}^2 \setminus \widetilde{\mathcal{N}})$. Let $\hat{Z}$ denote the mapping on $\mathbb{R}^2 \setminus \widetilde{\mathcal{N}}$ given by $(x,y) \mapsto (\hat{X}(x,y), \hat{Y}(x,y))$. Since, for any $t \in \mathbb{R} \setminus \mathcal{N}$,

$$(\det J_{\hat{Z}})(-t, 0) = w'(t)^2 + (1 + \mathcal{C}w'(t))^2 \neq 0,$$

where $J_{\hat{Z}}$ is the Jacobian matrix associated to $\hat{Z}$, the Inverse Function Theorem shows that the restriction of $\hat{Z}$ to any sufficiently small neighbourhood Π_t of $(-t, 0)$ is a diffeomorphism of class $C^{n+1,\alpha}$ onto a neighbourhood Ξ_t of the point $(-(\gamma + t + \mathcal{C}w(t)), w(t))$ on $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$. Since Z is a global bijection from $\mathbb{R}^2_-$ onto Ω, it is not difficult to see that Π_t and Ξ_t can be chosen so that $\hat{Z}$ maps $\Pi_t \cap \mathbb{R}^2_-$ onto $\Xi_t \cap \Omega$ and $\Pi_t \cap \mathbb{R}^2_+$ onto $\Xi_t \cap (\mathbb{R}^2 \setminus \overline{\Omega})$. If we consider on Ξ_t the inverse $\hat{\omega}_t = (\hat{\varphi}_t, \hat{\psi}_t)$ of $\hat{Z}$, then clearly $\hat{\psi}_t$ is a local $C^{n+1,\alpha}$ extension of ψ across $\mathcal{S}$. This shows that $\psi \in C^{n+1,\alpha}_{\mathrm{loc}}(\Omega \cup (\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}))$ and that (1.1j) holds in the form

$$|\nabla\psi|^2 = \lambda(Y) \quad \text{everywhere on } \mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}. \tag{2.2}$$

It is now legitimate to differentiate (1.1i) with respect to t to get that the tangential derivative of ψ is zero on $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$, and hence to conclude that (1.1j ′) holds in the form

$$\frac{\partial\psi}{\partial n} = h(Y) \quad \text{everywhere on } \mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}. \tag{2.3}$$

The preceding considerations can be summarized in the following result.

Theorem 2.2. *Let* $(\mathcal{S}, \psi)$ *satisfy* (1.1), *where* $h \in C^{n,\alpha}_{\mathrm{loc}}(-\infty, 0)$ *for some* $n \in \mathbb{N} \cup \{0\}$ *and* $\alpha \in (0, 1)$. *Then* $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$ *is a* $C^{n+1,\alpha}_{\mathrm{loc}}$ *curve,* $\psi \in C^{n+1,\alpha}_{\mathrm{loc}}(\Omega \cup (\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}))$ *and* (2.2), (2.3) *hold.*

2.2 Geometric Properties

2.2.1 Preliminaries

Proposition 2.3. *Let* $(\mathcal{S}, \psi)$ *be a solution of* (1.1a)-(1.1i). *Then there exists* $d \in \mathbb{R}$ *such that*

$$\psi(X, Y) - Y - d \to 0 \quad \textit{as } Y \to -\infty, \textit{ uniformly in } X. \tag{2.4}$$

Proof of Proposition 2.3. With ω and Z as in the previous chapter, it was proved in [27] that the function V given by (1.20) has a removable singularity at 0. Let $V(0) = -c - id$. It follows that

$$Z(x + iy) - (x + iy) \to -c - id \quad \text{as } y \to -\infty, \text{ uniformly in } x.$$

Therefore

$$\omega(X+iY)-(X+iY)\to c+id \quad \text{as } Y\to-\infty, \text{ uniformly in } X,$$

and hence, in particular, (2.4) holds. $\square$

Note that, if there exists $Y_0 \in (-\infty, 0)$ such that $h(Y_0) = 1$, then $\mathcal{S}_0 := \{(X, Y_0) : X \in \mathbb{R}\}$ and $\psi_0(X,Y) := Y - Y_0$ in the domain below $\mathcal{S}_0$ yields a solution of (1.1), which we call a trivial solution.

Proposition 2.4. *Let $h \in C^{0,\alpha}_{\mathrm{loc}}(-\infty, 0)$, and let $(\mathcal{S}, \psi)$ be a non-trivial solution of* (1.1). *Let $Y_c := \max\{v(s) : s \in \mathbb{R}\}$ and $Y_t := \min\{v(s) : s \in \mathbb{R}\}$. Then $h(Y_c) < 1$ and $h(Y_t) > 1$.*

Corollary 2.5. *In the notation of Proposition 2.4, there exists $Y_0 \in (Y_t, Y_c)$ such that $h(Y_0) = 1$.*

Proof of Proposition 2.4. Consider the function ξ in Ω given by

$$\xi(X,Y) := \psi(X,Y) - Y \quad \text{for all} \quad (X,Y) \in \Omega.$$

Then ξ is a harmonic function which, by Proposition 2.3, is bounded in Ω . Since $\psi = 0$ on $\mathcal{S}$, we conclude that $-Y_c \le \xi \le -Y_t$ on $\mathcal{S}$ and therefore, by the Maximum Principle, $-Y_c < \xi < -Y_t$ in Ω.

Let X_c and X_t be such that $(X_c, Y_c) \in \mathcal{S}$ and $(X_t, Y_t) \in \mathcal{S}$. Then ξ attains its extrema in $\overline{\Omega}$ at these points. $\mathcal{S}$ has a horizontal tangent at the point (X_t, Y_t), and the same is true at (X_c, Y_c) if $Y_c < 0$. Moreover, by Theorem 2.2, $\mathcal{S}$ and ξ are smooth enough at these points so that Hopf Boundary-Point Lemma can be applied. We conclude that

$$0 < \xi_Y(X_t, Y_t) = \psi_Y(X_t, Y_t) - 1,$$

and, if $Y_c < 0$, then

$$0 > \xi_Y(X_c, Y_c) = \psi_Y(X_c, Y_c) - 1.$$

Hence (2.3) yields that $h(Y_t) > 1$, and $h(Y_c) < 1$ if $Y_c < 0$.

Note also that, if $Y_c = 0$, then one can see directly that $h(Y_c) = 0 < 1$. This completes the proof of Proposition 2.4. □

2.2.2 A Collection of Related Results

Throughout the rest of this section we assume that

$$h \in C^{1,\alpha}_{\text{loc}}(-\infty, 0) \qquad \text{for some } \alpha \in (0,1). \tag{2.5}$$

As seen in Chapter 1, solutions w of (1.25) give rise to solutions $(\mathcal{S}, \psi)$ of (1.1) only if (1.24) holds, and then $\mathcal{S}$ is given in parametric form by (1.23). It is therefore of interest to exhibit situations when (1.24) follows automatically from (1.25). Recall that if (1.25) holds, then functions τ, ϑ can be defined by (1.29), and (1.30) holds.

Under the assumption that

$$h \text{ is decreasing and } \log h \text{ is concave on } (-\infty, 0), \tag{2.6}$$

it was proved in [27, Theorem 2.4] that solutions w of (1.25) for which $\mathcal{N}$ is at most denumerable satisfy

$$\vartheta(t) \in (-\pi/2, \pi/2) \qquad \text{for almost all } t \in \mathbb{R}, \tag{2.7}$$

and hence

$$1 + \mathcal{C}w'(t) > 0 \qquad \text{for almost all } t \in \mathbb{R}. \tag{2.8}$$

By (1.30), (2.8) ensures that (1.24) holds and, moreover, $\mathcal{S}$ is globally a graph, in the sense that there exists a continuous 2π-periodic function $\eta : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{S} = \{(X, \eta(X)) : \ X \in \mathbb{R}\}. \tag{2.9}$$

Here the main result is that, under the assumption (2.6), free boundaries must necessarily be global graphs, irrespective of the number of stagnation points.

Theorem 2.6. *Suppose that h satisfies* (2.6)*, and let $(\mathcal{S}, \psi)$ be a solution of* (1.1)*. Then $\mathcal{S}$ is of the form* (2.9)*, for some continuous 2π-periodic function $\eta : \mathbb{R} \to \mathbb{R}$.*

Theorem 2.6 is an immediate consequence of the following result.

Theorem 2.7. *Suppose that h satisfies* (2.6). *Let w be a solution of* (1.25) *such that* (1.24) *holds. Then w satisfies* (2.8).

The important point to note is that the assumptions of Theorem 2.6 and Theorem 2.7 admit the possibility that the set of stagnation points might be non-denumerable. This was not the case in [27, Theorem 2.4], where the denumerability of $\mathcal{N}$ was crucial. On the other hand, in [27, Theorem 2.4], the fact that (1.24) holds is not an assumption, but a conclusion. One can therefore say that Theorem 2.7 is neither weaker nor stronger than, but rather complementary to, [27, Theorem 2.4].

The next result is a local version of Theorem 2.7.

Theorem 2.8. *Suppose that there exists $\delta \in (-\infty, 0)$ such that*

$$h \text{ is decreasing and } \log h \text{ is concave on } [\delta, 0). \tag{2.10}$$

Let w be a solution of (1.25) *which satisfies* (1.24) *and for which $\mathcal{N} \neq \emptyset$. Then there exists $\beta \in [\delta, 0)$ which depends only on $\min \mathcal{R}(w)$, such that if $w(t) \in [\beta, 0]$ for all $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathcal{N}$ and $(t_1, t_2) \subset \mathbb{R} \setminus \mathcal{N}$, then*

$$1 + Cw'(t) > 0 \quad \text{for all } t \in (t_1, t_2). \tag{2.11}$$

Theorem 2.8 is particularly relevant when there are infinitely many stagnation points on a period of w. Indeed, if $a \in \mathcal{N}$ and if $[a, a + 2\pi] \setminus \mathcal{N} = \cup_{i=1}^{\infty}(a_i, b_i)$, with $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$, then the fact that w is a function of bounded variation shows that Theorem 2.8 is applicable on all but finitely many intervals (a_i, b_i).

The next result can be used to deduce another geometric property of free boundaries.

Proposition 2.9. *Suppose that h satisfies* (2.6) *and w satisfies* (1.25). *Let s_1, s_2 with $s_1 < s_2$ be such that $h(w(s_1)) = h(w(s_2)) = 1$ and $h(w(s)) > 1$ for all $s \in (s_1, s_2)$. Then $\vartheta' \geq 0$ on (s_1, s_2).*

Indeed, suppose that $\mathcal{S}$ is of the form (2.9). Let $Y_0 \in (Y_t, Y_c)$ be such that $h(Y_0) = 1$, which exists by Corollary 2.5. Proposition 2.9 shows that, if X_1,

X_2 with $X_1 < X_2$ are such that $\eta(X_1) = \eta(X_2) = Y_0$ and $\eta(X) < Y_0$ for all $X \in (X_1, X_2)$ then η is a convex function on the interval (X_1, X_2).

The following result provides bounds for solutions of (1.25) when (2.6) holds.

Theorem 2.10. *Suppose that h satisfies* (2.6)*, and let w be a solution of* (1.25) *for which* (2.7) *holds. Let t_1, t_2 be such that $(t_1, t_2) \subset \mathbb{R} \setminus \mathcal{N}$.*

(i) Suppose that t_1, $t_2 \in \mathbb{R} \setminus \mathcal{N}$, $w'(t_1) = w'(t_2) = 0$, and either $0 \not\equiv w' \geq 0$ on (t_1, t_2), or $0 \not\equiv w' \leq 0$ on (t_1, t_2). Then

$$h(w(t_1)) + h(w(t_2)) \leq 2. \tag{2.12}$$

(ii) Suppose that $t_1 \in \mathbb{R} \setminus \mathcal{N}$, $t_2 \in \mathcal{N}$, $w'(t_1) = 0$, $0 \not\equiv w' \geq 0$ on (t_1, t_2). Then $h(w(t_1)) \leq 2$.

(iii) Suppose that $t_1 \in \mathcal{N}$, $t_2 \in \mathbb{R} \setminus \mathcal{N}$, $w'(t_1) = 0$, $0 \not\equiv w' \leq 0$ on (t_1, t_2). Then $h(w(t_2)) \leq 2$.

Note that in Theorem 2.10 the requirement that (2.7) holds is an extremely weak restriction when h is decreasing and $\log h$ is concave, since by [27, Theorem 2.4] and Theorem 2.7 it is satisfied automatically for all solutions w of (1.25) with at most countably many stagnation points, as well as for those, with any number of stagnation points, which give rise to free boundaries.

Corollary 2.11. *Suppose that h satisfies* (2.6)*, and let w be a solution of* (1.25) *for which* (2.7) *holds. Then $h(w(t)) \leq 2$ for all $t \in \mathbb{R}$.*

In terms of free boundaries, Theorem 2.10 can be interpreted as follows. Let $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic. If X_1, X_2 are such that $\eta'(X_1) = \eta'(X_2) = 0$ and $\eta'(X) \neq 0$ for all $X \in (X_1, X_2)$, then

$$|\nabla\psi|(X_1, \eta(X_1)) + |\nabla\psi|(X_2, \eta(X_2)) \leq 2. \tag{2.13}$$

Corollary 2.11 shows that $|\nabla\psi| \leq 2$ everywhere on $\mathcal{S}$, and hence by the Maximum Principle

$$|\nabla\psi| \leq 2 \quad \text{everywhere in } \Omega. \tag{2.14}$$

It is remarkable that the estimates (2.13) and (2.14) are independent of the nonlinearity h, provided that it satisfies (2.6). For a restricted class of solutions of the water-wave problem, the estimate (2.13) was noted in passing by Toland [33, (x), p. 15], by developing some arguments of Spielvogel [28].

We now give some new results in the case when there exists a compact interval $J \subset (-\infty, 0)$ such that

$$h \text{ is decreasing and } \log h \text{ is convex on } J. \tag{2.15}$$

The following result is an analogue of Theorem 2.10 when (2.15) holds instead of (2.6).

Theorem 2.12. *Suppose that h satisfies* (2.15), *and let w be a solution of* (1.25) *with $\mathcal{R}(w) \subseteq J$, and for which* (2.7) *holds. Let t_1, t_2 with $t_1 < t_2$ be such that $w'(t_1) = w'(t_2) = 0$, and either $0 \not\equiv w' \geq 0$ on (t_1, t_2), or $0 \not\equiv w' \leq 0$ on (t_1, t_2). Then*

$$h(w(t_1)) + h(w(t_2)) \geq 2. \tag{2.16}$$

Corollary 2.13. *Suppose that h satisfies* (2.15), *and let w be a solution of* (1.25) *with $\mathcal{R}(w) \subseteq J$, and for which* (2.7) *holds. Let s_1, s_2 with $s_1 < s_2$ be such that $h(w(s_1)) = h(w(s_2)) = 1$ and $h(w(s)) < 1$ for all $s \in (s_1, s_2)$. Then w does not have any strict local minima in (s_1, s_2).*

The interpretation of Theorem 2.12 and Corollary 2.13 in terms of free boundaries is obvious.

We now show, by means of a family of examples of explicit solutions of (1.25), that the results of Theorem 2.10, Corollary 2.11 and Theorem 2.12 are sharp, at least for non-singular solutions. Indeed, it is shown in [27, Subsection 1.6.1] that, if $\widetilde{\lambda}_b : \mathbb{R} \to (0, \infty)$, where $0 \leq b < 1$, is given by $\widetilde{\lambda}_b(r) = (1+b^2)e^{-2r}$ for all $r \in \mathbb{R}$, and if

$$\widetilde{w}_b(t) = -\frac{1}{2} \log \Big(\frac{1 + b^2 + 2b \sin t}{1 + b^2} \Big), \tag{2.17}$$

then $\widetilde{w}_b$ is a non-singular solution of (1.25) with $\lambda = \widetilde{\lambda}_b$. Note that $\widetilde{\lambda}_b$ is decreasing and $\log \widetilde{\lambda}_b$ is affine on $\mathbb{R}$, but $\widetilde{\lambda}_b$ does not satisfy (1.2). To overcome this inconvenience, let $a > \max \mathcal{R}(w)$ and let $\widehat{\lambda}_b : (-\infty, 0] \to [0, \infty)$ be such that (1.2) holds, $\widehat{\lambda}_b \in C^{1,\alpha}_{\text{loc}}(-\infty, 0)$, and

$$\widehat{\lambda}_b(r) = \widetilde{\lambda}_b(r + a) \text{ for all } r \in (-\infty, \max \mathcal{R}(w) - a],$$
$$\widehat{\lambda}_b \text{ is decreasing and } \log \widehat{\lambda}_b \text{ is concave on } (-\infty, 0).$$

Let also $\widehat{w}_b := \widetilde{w}_b - a$. It is straightforward that $\widehat{w}_b$ is a non-singular solution of (1.25) with $\lambda = \widehat{\lambda}_b$. With $\widehat{h}_b := \sqrt{\widehat{\lambda}_b}$ and $J := [\min \mathcal{R}(\widetilde{w}_b) - a, \max \mathcal{R}(\widetilde{w}_b) - a]$, $\widehat{h}_b$ satisfies the assumptions of both Theorem 2.10 and Theorem 2.12. Note that, for all $t \in \mathbb{R}$,

$$\widehat{\lambda}_b(\widehat{w}_b(t)) = \widetilde{\lambda}_b(\widetilde{w}_b(t)) = 1 + b^2 + 2b\sin t, \tag{2.18}$$

and $\widehat{w}_b{}'(t) = 0$ if and only if $t = n\pi + \pi/2$ for some $n \in \mathbb{Z}$. It is immediate from (2.18) that, for all $n \in \mathbb{Z}$,

$$\widehat{h}_b(\widehat{w}_b(n\pi - \pi/2)) + \widehat{h}_b(\widehat{w}_b(n\pi + \pi/2)) = 2,$$

hence the results of Theorem 2.10 and Theorem 2.12 are sharp. Also, since

$$\max\{\widehat{h}_b(\widehat{w}_b(t) : t \in \mathbb{R}\} = 1 + b \quad \text{for all } b \in [0,1),$$

it follows that the result of Corollary 2.11 is also sharp.

We conclude this Subsection with a result giving a sufficient condition for solutions of (1.25) to give rise to free boundaries when h satisfies (2.15).

Theorem 2.14. *Suppose that h satisfies* (2.15), *and let w be a solution of* (1.25) *with $\mathcal{R}(w) \subseteq J$, and such that $h(w(t)) \leq 2$ for all $t \in \mathbb{R}$. Then* (2.7) *holds, and hence so does* (2.8).

2.2.3 Proofs of the Results in Subsection 2.2.2

We now give the proofs of the results in the previous subsection. We start with general consideration which are relevant for all the proofs.

If I is any open interval contained in $\mathbb{R} \setminus \mathcal{N}$ then, since (2.5) holds, it follows from Theorem 2.1 that $w \in C^{2,\alpha}_{\mathrm{loc}}(I)$. It is proved in [27] that

$$\begin{aligned}&(\mathcal{C}\log h(w))'(t) - \frac{h'(w(t))}{h(w(t))}\mathcal{C}w'(t)\\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\log h(w(t)) + (h'(w(t)/h(w(t))(w(s) - w(t)) - \log h(w(s))}{\sin^2 \frac{t-s}{2}}\, ds. \end{aligned}\tag{2.19}$$

The formula (2.19), which is just an identity based on the definition of a conjugate function, is valid for all $t \in I$ under the present regularity assumptions on h and

w. Since $I \subset \mathbb{R} \setminus \mathcal{N}$ was arbitrary, (2.19) is valid everywhere on $\mathbb{R} \setminus \mathcal{N}$.

One can now see that, if w is a solution of (1.25), then $\vartheta \in C^{1,\alpha}_{\rm loc}(\mathbb{R} \setminus \mathcal{N})$ and, for all $t \in \mathbb{R} \setminus \mathcal{N}$,

$$\vartheta'(t) - \frac{h'(w(t))}{h(w(t))}\left(\frac{\cos\vartheta(t)}{h(w(t))} - 1\right) \geq 0 \quad \text{if } \log h \text{ is concave,} \tag{2.20}$$

and

$$\vartheta'(t) - \frac{h'(w(t))}{h(w(t))}\left(\frac{\cos\vartheta(t)}{h(w(t))} - 1\right) \leq 0 \quad \text{if } \log h \text{ is convex.} \tag{2.21}$$

The estimate (2.20) is one of the key ingredients in the proof of [27, Theorem 2.4]. See also [36] for smooth solutions of the water-wave problem.

Proof of Proposition 2.9. Since h is decreasing and, for $s \in (s_1, s_2)$, $\cos\vartheta(s) < h(w(s))$, it is immediate from (2.20) that $\vartheta' \geq 0$ on (s_1, s_2). □

At the heart of the proofs of Theorem 2.10, Theorem 2.12 and Theorem 2.14 lies the following new identity satisfied by solutions of (1.25) everywhere on $\mathbb{R}\setminus\mathcal{N}$:

$$\begin{aligned}\frac{d}{dt}\left(\frac{h^2(w)}{2} - h(w)\cos\vartheta\right) &= h(w)h'(w)w' - h'(w)w'\cos\vartheta + h(w)\sin\vartheta\;\vartheta' \\ &= w'h^2(w)\left\{\vartheta' - \frac{h'(w)}{h(w)}\left(\frac{\cos\vartheta}{h(w)} - 1\right)\right\}\end{aligned} \tag{2.22}$$

Proof of Theorem 2.10. Suppose first that we are in the situation (i), and with $0 \not\equiv w' \geq 0$, on (t_1, t_2). It follows from (2.22), using (2.20), that

$$t \mapsto \left(\frac{h^2(w(t))}{2} - h(w(t))\cos\vartheta(t)\right) \quad \text{is increasing on } (t_1, t_2). \tag{2.23}$$

Since $\vartheta \in (-\pi/2, \pi/2)$, it follows that $\vartheta(t_1) = \vartheta(t_2) = 0$. Hence we deduce from (2.23) that

$$\frac{h^2(w(t_1))}{2} - h(w(t_1)) \leq \frac{h^2(w(t_2))}{2} - h(w(t_2)),$$

or, equivalently,

$$\big(h(w(t_1)) - h(w(t_2))\big)\big(h(w(t_1)) + h(w(t_2)) - 2\big) \leq 0. \tag{2.24}$$

But $w(t_1) < w(t_2)$ by assumption, and hence, since h is strictly decreasing,

w. Since $I \subset \mathbb{R} \setminus \mathcal{N}$ was arbitrary, (2.19) is valid everywhere on $\mathbb{R} \setminus \mathcal{N}$.

One can now see that, if w is a solution of (1.25), then $\vartheta \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R} \setminus \mathcal{N})$ and, for all $t \in \mathbb{R} \setminus \mathcal{N}$,

$$\vartheta'(t) - \frac{h'(w(t))}{h(w(t))}\left(\frac{\cos\vartheta(t)}{h(w(t))} - 1\right) \geq 0 \quad \text{if } \log h \text{ is concave,} \tag{2.20}$$

and

$$\vartheta'(t) - \frac{h'(w(t))}{h(w(t))}\left(\frac{\cos\vartheta(t)}{h(w(t))} - 1\right) \leq 0 \quad \text{if } \log h \text{ is convex.} \tag{2.21}$$

The estimate (2.20) is one of the key ingredients in the proof of [27, Theorem 2.4]. See also [36] for smooth solutions of the water-wave problem.

Proof of Proposition 2.9. Since h is decreasing and, for $s \in (s_1, s_2)$, $\cos\vartheta(s) < h(w(s))$, it is immediate from (2.20) that $\vartheta' \geq 0$ on (s_1, s_2). □

At the heart of the proofs of Theorem 2.10, Theorem 2.12 and Theorem 2.14 lies the following new identity satisfied by solutions of (1.25) everywhere on $\mathbb{R} \setminus \mathcal{N}$:

$$\begin{aligned}\frac{d}{dt}\left(\frac{h^2(w)}{2} - h(w)\cos\vartheta\right) &= h(w)h'(w)w' - h'(w)w'\cos\vartheta + h(w)\sin\vartheta\ \vartheta' \\ &= w'h^2(w)\left\{\vartheta' - \frac{h'(w)}{h(w)}\left(\frac{\cos\vartheta}{h(w)} - 1\right)\right\}\end{aligned} \tag{2.22}$$

Proof of Theorem 2.10. Suppose first that we are in the situation (i), and with $0 \not\equiv w' \geq 0$, on (t_1, t_2). It follows from (2.22), using (2.20), that

$$t \mapsto \left(\frac{h^2(w(t))}{2} - h(w(t))\cos\vartheta(t)\right) \quad \text{is increasing on } (t_1, t_2). \tag{2.23}$$

Since $\vartheta \in (-\pi/2, \pi/2)$, it follows that $\vartheta(t_1) = \vartheta(t_2) = 0$. Hence we deduce from (2.23) that

$$\frac{h^2(w(t_1))}{2} - h(w(t_1)) \leq \frac{h^2(w(t_2))}{2} - h(w(t_2)),$$

or, equivalently,

$$\big(h(w(t_1)) - h(w(t_2))\big)\big(h(w(t_1)) + h(w(t_2)) - 2\big) \leq 0. \tag{2.24}$$

But $w(t_1) < w(t_2)$ by assumption, and hence, since h is strictly decreasing,

$h(w(t_1)) > h(w(t_2))$. The required conclusion follows from (2.24).

Suppose now that we are in the situation (i), but with $0 \not\equiv w' \leq 0$ on (t_1, t_2). Instead of (2.23) we now have

$$t \mapsto \left(\frac{h^2(w(t))}{2} - h(w(t)) \cos \vartheta(t) \right) \quad \text{is decreasing on } (t_1, t_2), \tag{2.25}$$

and the required conclusion follows by an argument similar to that of the previous case.

Suppose now that we are in situation (ii). It follows as above that, for all $t \in (t_1, t_2)$,

$$\frac{h^2(w(t_1))}{2} - h(w(t_1)) \leq \frac{h^2(w(t))}{2} - h(w(t)) \cos \vartheta(t). \tag{2.26}$$

The required conclusion is obtained by passing to the limit in (2.26) as $t \nearrow t_2$, using the fact that $h(w(t)) \to 0$ as $t \nearrow t_2$, since $t_2 \in \mathcal{N}$.

The analysis in the situation (iii) is entirely analogous. This completes the proof of Theorem 2.10. $\square$

Proof of Corollary 2.11. Recall that $[\tau] = 0$, where $\tau = -\log h(w)$. If $h(w) \equiv c$, where $c > 0$ is a constant, then necessarily $c = 1$. Suppose that $h(w) \not\equiv c$. Let $t_1 \in \mathbb{R}$ be such that

$$w(t_1) = \min\{w(t) : t \in \mathbb{R}\}. \tag{2.27}$$

Then $w'(t_1) = 0$ and $h(w(t_1)) > 1$. Since h is decreasing, it suffices to prove that $h(w(t_1)) \leq 2$. Let $\hat{t} > t_1$ be such that $h(w(\hat{t})) = 1$ and $h(w(t)) > 1$ for all $t \in (t_1, \hat{t})$. By Proposition 2.9, $\vartheta'(t) \geq 0$ for all $t \in (t_1, \hat{t})$. Since $\vartheta(t_1) = 0$, it follows that $\vartheta(t) \geq 0$ for all $t \in (t_1, \hat{t})$, and therefore $0 \not\equiv w' \geq 0$ on $(t_1, \hat{t})$. There exists $t_2 > t_1$ with $(t_1, t_2) \subset \mathbb{R} \setminus \mathcal{N}$ such that $w' \geq 0$ on (t_1, t_2), and either $t_2 \in \mathbb{R} \setminus \mathcal{N}$ and $w'(t_2) = 0$, or $t_2 \in \mathcal{N}$. In either case we conclude from Theorem 2.10 that $h(w(t_1)) \leq 2$, which is the required result. $\square$

Proof of Theorem 2.12. It suffices to consider the case when $0 \not\equiv w' \geq 0$ on (t_1, t_2), since the other case can be treated in an entirely similar way. It follows from (2.22) using (2.21) that

$$t \mapsto \left(\frac{h^2(w(t))}{2} - h(w(t)) \cos \vartheta(t) \right) \quad \text{is decreasing on } (t_1, t_2),$$

so that

$$\frac{h^2(w(t_1))}{2} - h(w(t_1)) \geq \frac{h^2(w(t_2))}{2} - h(w(t_2)),$$

or, equivalently,

$$\big(h(w(t_1)) - h(w(t_2))\big)\big(h(w(t_1)) + h(w(t_2)) - 2\big) \geq 0.$$

Since $w(t_1) < w(t_2)$ and h is strictly decreasing, the required conclusion follows. $\square$

Proof of Corollary 2.13. Suppose for a contradiction that $s_0 \in (s_1, s_2)$ is a strict local minimum of w. Then $w'(s_0) = 0$. Let $\hat{s} \in (s_0, s_2)$ be such that

$$w(\hat{s}) = \max\{w(s) : s \in [s_0, s_2]\}.$$

Then there exist t_1, t_2 with $s_0 \leq t_1 < t_2 \leq \hat{s}$, such that $w'(t_1) = w'(t_2) = 0$ and $0 \not\equiv w' \geq 0$ on (t_1, t_2). Now Theorem 2.12 yields that $h(w(t_1)) + h(w(t_2)) \geq 2$, which contradicts the fact that $h(w(t_1)) < 1$ and $h(w(t_2)) < 1$. Hence no strict local minima of w on (s_1, s_2) exist. $\square$

Proof of Theorem 2.14. Since ϑ is continuous, 2π-periodic, and has zero mean, $\mathcal{R}(\vartheta)$ is a compact interval which contains 0. We shall prove that $\pi/2 \notin \mathcal{R}(\vartheta)$ and $-\pi/2 \notin \mathcal{R}(\vartheta)$.

Suppose that $\pi/2 \in \mathcal{R}(\vartheta)$. Then there exist t_1, t_2 with $t_1 < t_2$ such that $\vartheta(t_1) = 0$, $\vartheta(t_2) = \pi/2$, and $0 < \vartheta < \pi/2$ on (t_1, t_2). It follows from (2.22) using (2.21) that

$$t \mapsto \left(\frac{h^2(w(t))}{2} - h(w(t)) \cos\vartheta(t)\right) \quad \text{is decreasing on } (t_1, t_2),$$

so that

$$\frac{h^2(w(t_1))}{2} - h(w(t_1)) \geq \frac{h^2(w(t_2))}{2} > 0.$$

But this contradicts the assumption that $h(w(t)) \leq 2$ for all $t \in \mathbb{R}$. Hence $\pi/2 \notin \mathcal{R}(\vartheta)$.

Suppose now that $-\pi/2 \in \mathcal{R}(\vartheta)$. Then there exist t_1, t_2 with $t_1 < t_2$ such that $\vartheta(t_1) = -\pi/2$, $\vartheta(t_2) = 0$, and $-\pi/2 < \vartheta < 0$ on (t_1, t_2). It follows from

(2.22) using (2.21) that

$$t \mapsto \left(\frac{h^2(w(t))}{2} - h(w(t))\cos\vartheta(t)\right) \quad \text{is increasing on } (t_1, t_2),$$

so that

$$0 < \frac{h^2(w(t_1))}{2} \leq \frac{h^2(w(t_2))}{2} - h(w(t_2)),$$

which again contradicts the assumption that $h(w(t)) \leq 2$ for all $t \in \mathbb{R}$. Hence $-\pi/2 \notin \mathcal{R}(\vartheta)$. This completes the proof of Theorem 2.14. □

The following Lemma will be used in the proof of Theorem 2.7.

Lemma 2.15. *Let $\mathcal{T} := \{(u(s), v(s)) : s \in \mathbb{R}\}$ be a non-self-intersecting curve, where u, $v : \mathbb{R} \to \mathbb{R}$ are continuous functions, $v(s) \leq 0$ for all $s \in \mathbb{R}$, and*

$$s \mapsto (u(s) - s, v(s)) \quad \textit{is } 2\pi\textit{-periodic.}$$

Let $\mathcal{N} := \{s \in \mathbb{R} : v(s) = 0\}$. Then $u(s_1) < u(s_2)$ for every s_1, $s_2 \in \mathcal{N}$ with $s_1 < s_2$.

Proof of Lemma 2.15. It suffices to prove the required result when s_1, s_2 satisfy the additional assumption that $s_2 < s_1 + 2\pi$, for then the general case would follow immediately. For every a, $b \in \mathbb{R}$ with $a < b$, let us denote $\mathcal{T}_{a,b} := \{(u(s), v(s)) : s \in [a, b]\}$.

We argue by contradiction and assume that there exist s_1, $s_2 \in \mathcal{N}$ with $s_1 < s_2 < s_1 + 2\pi$ such that $u(s_1) > u(s_2)$. Then $\mathcal{T}_{s_2, s_1+2\pi}$ is a non-self-intersecting curve contained in the closed lower half-plane, with endpoints at $(u(s_2), 0)$ and $(u(s_1) + 2\pi, 0)$. Note also that

$$u(s_2) < u(s_1) < u(s_1) + 2\pi = u(s_1 + 2\pi).$$

Let $\mathcal{B}$ be the continuous curve obtained as the union of the following curves: $\mathcal{T}_{s_2, s_1+2\pi}$, the vertical line segment joining the points $(u(s_1) + 2\pi, 0)$ and $(u(s_1) + 2\pi, 1)$, the horizontal line segment joining $(u(s_1) + 2\pi, 1)$ and $(u(s_2), 1)$, the vertical line segment joining $(u(s_2), 1)$ and $(u(s_2), 0)$. Then $\mathcal{B}$ is a non-self-intersecting closed curve, i.e. a Jordan curve. By the Jordan Curve Theorem, the complement of $\mathcal{B}$ has exactly two connected components, one bounded and one unbounded.

Since the point $(u(s_1), 0)$ does not belong to $\mathcal{B}$ and is the limit of the sequence of points $\{(u(s_1), 1/n)\}_{n\geq 1}$, all of which clearly belong to the bounded component of $\mathbb{C}\backslash\mathcal{B}$, we deduce that $(u(s_1), 0)$ belongs to the bounded component of $\mathbb{C}\backslash\mathcal{B}$. But for $k \in \mathbb{N}$ sufficiently large, the point $(u(s_1) - 2k\pi, 0)$ belongs to the unbounded component of $\mathbb{C} \setminus \mathcal{B}$, and can be joined to $(u(s_1), 0)$ by the curve $\mathcal{T}_{s_1-2k\pi,s_1}$. It follows that $\mathcal{T}_{s_1-2k\pi,s_1}$ must intersect $\mathcal{B}$. But this is impossible, since on the one hand $\mathcal{T}_{s_1-2k\pi,s_1}$ is contained in the closed lower half-plane, and on the other hand $\mathcal{T}_{s_1-2k\pi,s_1}$ and $\mathcal{T}_{s_2,s_1+2\pi}$ do not intersect, as $[s_1 - 2k\pi, s_1] \cap [s_2, s_1 + 2\pi] = \emptyset$ and $\mathcal{T}$ was assumed non-self-intersecting. The required conclusion follows. □

We also recall for easy reference some results of [27] which are valid under the assumptions of Theorem 2.7. Lemma 2.16 below is a restatement of [27, Lemma 3.11], while Lemma 2.17 is [27, Lenmma 3.12].

Lemma 2.16. *Suppose that $c \in (a, b) \subset \mathbb{R} \setminus \mathcal{N}$ and let $\ell_1, \ell_2 \in \mathbb{Z}$ be such that $\ell_1\pi + \pi/2 \leq \vartheta(c) \leq \ell_2\pi + \pi/2$. Then*

$$\vartheta(t) > \ell_1\pi + \pi/2 \textit{ for } t \in (c, b) \quad \textit{and} \quad \vartheta(t) < \ell_2\pi + \pi/2 \textit{ for } t \in (a, c).$$

Lemma 2.17. *Suppose that $a, b \in \mathcal{N}$ and $(a, b) \subset \mathbb{R}\backslash\mathcal{N}$. Then either $\vartheta(t) \to +\infty$ as $t \nearrow b$ or there exists $\varsigma > 0$ such that*

$$\cos\vartheta(t) > 0 \textit{ for } t \in (b - \varsigma, b).$$

Similarly, either $\vartheta(t) \to -\infty$ as $t \searrow a$ or there exists $\varsigma > 0$ such that

$$\cos\vartheta(t) > 0 \textit{ for } t \in (a, a + \varsigma).$$

The proof of Theorem 2.7 ultimately rests on an application of the following classical theorem in plane topology, see e.g. Krasnoselskii and al. [11, Theorem 2.3, p.16], to a suitably devised Jordan curve.

Theorem 2.18. *Let $\sigma : [a, b] \to \mathbb{C}$ be a parametrization of a Jordan curve, where σ is a function of class C^1, with $\sigma'(a) = \sigma'(b)$ and $|\sigma'| > 0$ on $[a, b]$. Let $\phi : [a, b] \to \mathbb{R}$ be a continuous function such that*

$$\sigma'(t) = |\sigma'(t)| \exp\{i\phi(t)\} \quad \textit{for all } t \in [a, b].$$

Then $\phi(b) - \phi(a)$ equals either 2π or -2π.

Proof of Theorem 2.7. Let w be a solution of (1.25) for which (1.24) holds. We only consider the more interesting and difficult case when $\mathcal{N} \neq \emptyset$. We aim to prove that, for every $t_1, t_2 \in \mathcal{N}$ with $(t_1, t_2) \subset \mathbb{R} \setminus \mathcal{N}$, one has that

$$1 + \mathcal{C}w'(t) > 0 \quad \text{for all } t \in (t_1, t_2). \tag{2.28}$$

Fix $t_1, t_2 \in \mathcal{N}$ with $(t_1, t_2) \subset \mathbb{R} \setminus \mathcal{N}$. Let

$$\widetilde{\mathcal{S}} := \{(t + \mathcal{C}w(t), w(t)) : t \in \mathbb{R}\}, \tag{2.29}$$

which by assumption is a non-self-intersecting curve. (The curve $\widetilde{\mathcal{S}}$ is obtained from the curve $\mathcal{S}$ given by (1.23) by reflection with respect to a vertical line.) Observe that, for all $t \in (t_1, t_2)$, $\vartheta(t)$ is the angle between the tangent to $\widetilde{\mathcal{S}}$ at $(t + \mathcal{C}w(t), w(t))$ and the horizontal. It follows from Lemma 2.15 that

$$t_1 + \mathcal{C}w(t_1) \; < \; t_2 + \mathcal{C}w(t_2).$$

Let $T_1 := t_1 + \mathcal{C}w(t_1)$, $T_2 := t_2 + \mathcal{C}w(t_2)$, so that $T_1 < T_2$.

To prove (2.29) we argue by contradiction and assume that there exists $t_0 \in (t_1, t_2)$ with $\cos\vartheta(t_0) \le 0$, which means

$$\vartheta(t_0) \in [2k\pi + \pi/2, 2k\pi + 3\pi/2] \quad \text{for some } k \in \mathbb{Z}. \tag{2.30}$$

By Lemma 2.16 and Lemma 2.17, there exist $\tilde{t}_1, \tilde{t}_2 \in (t_1, t_2)$ with $\tilde{t}_1 \le t_0 \le \tilde{t}_2$ such that

$$\begin{gathered}
\vartheta(\tilde{t}_1) = 2k\pi + \pi/2, \qquad\qquad \vartheta(\tilde{t}_2) = 2k\pi + 3\pi/2, \\
\vartheta(\tilde{t}_1) < \vartheta(t) < \vartheta(\tilde{t}_2) \;\text{ for all } t \in (\tilde{t}_1, \tilde{t}_2) \\
\vartheta(t) < \vartheta(\tilde{t}_1) \;\text{ for all } t \in (t_1, \tilde{t}_1), \qquad \vartheta(t) > \vartheta(\tilde{t}_2) \;\text{ for all } t \in (\tilde{t}_2, t_2).
\end{gathered} \tag{2.31}$$

Let $\gamma : \mathbb{R} \to \mathbb{C}$ be given by

$$\gamma(t) := t + \mathcal{C}w(t) + iw(t) \qquad \text{for all } t \in \mathbb{R}.$$

Fix $\epsilon > 0$ with $\epsilon < (T_2 - T_1)/2$. Let s_1, s_2 with $s_1 \in (t_1, \tilde{t}_1]$ and $s_2 \in [\tilde{t}_2, t_2)$ be such that

$$|\gamma(s) - T_1| \leq \epsilon \text{ for all } s \in (t_1, s_1], \qquad |\gamma(s) - T_2| \leq \epsilon \text{ for all } s \in [s_2, t_2). \quad (2.32)$$

Let $D < 0$ be given by

$$D := \max\{w(t) : t \in [s_1, s_2]\},$$

and let $p_1 \in (t_1, s_1]$ and $p_2 \in [s_2, t_2)$ be such that

$$w(p_1) = w(p_2) = D/2, \qquad w(t) < D/2 \quad \text{for all } t \in (p_1, p_2).$$

It follows that $w'(p_1) \leq 0$ and $w'(p_2) \geq 0$. Hence there exist k_1, $k_2 \in \mathbb{Z}$ such that

$$\vartheta(p_1) \in [(2k_1 - 1)\pi, 2k_1\pi], \qquad \vartheta(p_2) \in [2k_2\pi, (2k_2 + 1)\pi]. \quad (2.33)$$

It follows from (2.31) that

$$k_1 \leq k, \qquad k_2 \geq k + 1, \quad (2.34)$$

where k is as in (2.30). Let $P_1 := p_1 + Cw(p_1)$, $P_2 := p_2 + Cw(p_2)$. It follows from (2.32) that $P_1 < P_2$. Note also that both the points $\gamma(p_1)$ and $\gamma(p_2)$ lie on the horizontal line $Y = D/2$.

Let $\tilde{\gamma} : [p_1, p_2] \to \mathbb{C}$ be the restriction of γ to $[p_1, p_2]$. Fix $\varepsilon > 0$ with $\varepsilon < (P_2 - P_1)/2$, and fix $A \in \mathbb{R}$ with $A > D/2$. Recall the definition of the function $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$, which is

$$\operatorname{sgn} x = x/|x| \quad \text{if} \quad x \neq 0, \qquad \operatorname{sgn} 0 = 0.$$

It is obvious that, for some q_1, $q_2 \in \mathbb{R}$ with $q_1 < p_1$, $p_2 < q_2$, one can construct a function $\hat{\gamma} : [q_1, q_2] \to \mathbb{C}$, where

$$\hat{\gamma}(t) := u(q) + iv(q) \quad \text{for all } q \in [q_1, q_2],$$

such that $\hat{\gamma}$ is an extension of $\tilde{\gamma}$, and it has the following additional properties:

$$
\begin{aligned}
&u, v : [q_1, q_2] \to \mathbb{R} \quad \text{are of class } C^1, && (2.35\text{a})\\
&u'(q)^2 + v'(q)^2 > 0 \quad \text{for all } q \in [q_1, q_2], &&\\
&v(q_1) = v(q_2) = A, && (2.35\text{b})\\
&u(q_1) = u(p_1) - \varepsilon \operatorname{sgn}(1 + \mathcal{C}w'(p_1)), && (2.35\text{c})\\
&u(q_2) = u(p_2) + \varepsilon \operatorname{sgn}(1 + \mathcal{C}w'(p_2)), && (2.35\text{d})\\
&v'(q_1) = -1, \qquad v' \le 0 \text{ on } [q_1, p_1], && (2.35\text{e})\\
&v'(q_2) = 1, \qquad v' \ge 0 \text{ on } [p_2, q_2], && (2.35\text{f})\\
&u'(q_1) = 0, \qquad \operatorname{sgn} u' = \operatorname{sgn}(1 + \mathcal{C}w'(p_1)) \text{ on } (q_1, p_1], && (2.35\text{g})\\
&u'(q_2) = 0, \qquad \operatorname{sgn} u' = \operatorname{sgn}(1 + \mathcal{C}w'(p_2)) \text{ on } (p_2, q_2]. && (2.35\text{h})
\end{aligned}
$$

If we denote $Q_1 := u(q_1)$, $Q_2 := u(q_2)$, then $Q_1 < Q_2$. Now consider, on the circle having the segment joining (Q_1, A) and (Q_2, A) as a diameter, the semi-circle $\mathcal{B}$ joining these two points and containing the point $((Q_1+Q_2)/2, A+(Q_2-Q_1)/2)$. Let $Z_0 := (Q_1 + Q_2)/2 + i[A + (Q_2 - Q_1)/2]$.

Let r_1, r_2 with $r_1 < q_1$, $r_2 > q_2$ and consider a C^1 function $\gamma_* : [r_1, r_2] \to \mathbb{C}$ which is an extension of $\hat{\gamma}$, such that

$$
\begin{aligned}
&\gamma_*(r_1) = \gamma_*(r_2) = Z_0, && (2.36\text{a})\\
&\gamma_*'(r_1) = \gamma_*'(r_2) = -1 + i0, && (2.36\text{b})\\
&|\gamma_*'(r)| > 0 \quad \text{for all } r \in [r_1, r_2], && (2.36\text{c})
\end{aligned}
$$

and

$\gamma_*|_{[r_1,q_1]}$ is an injective parametrization of the arc of $\mathcal{B}$ joining Z_0 and $\hat{\gamma}(q_1)$,

$\gamma_*|_{[q_2,r_2]}$ is an injective parametrization of the arc of $\mathcal{B}$ joining $\hat{\gamma}(r_2)$ and Z_0.

It is very easy to prove that $\gamma_* : [r_1, r_2] \to \mathbb{R}$ constructed above provides a parametrization of a Jordan curve which has a continuously varying tangent. Let us write

$$\gamma_*'(r) = |\gamma_*'(r)| \exp\{i\vartheta_*(r)\} \quad \text{for all } r \in [r_1, r_2],$$

where $\vartheta_* : [r_1, r_2] \to \mathbb{R}$ is a continuous function which extends $\vartheta : [p_1, p_2] \to \mathbb{R}$.

It follows from (2.33) and (2.35) that

$$\vartheta_*(q_1) = 2k_1\pi - \pi/2, \qquad \vartheta_*(q_2) = 2k_2\pi + \pi/2.$$

Using (2.36) we deduce that

$$\vartheta_*(r_1) = 2k_1\pi - \pi, \qquad \vartheta_*(r_2) = 2k_2\pi + \pi.$$

Therefore

$$\vartheta_*(r_2) - \vartheta_*(r_1) = 2(k_2 - k_1)\pi + 2\pi, \tag{2.37}$$

where, by (2.34),

$$k_2 - k_1 \geq 1. \tag{2.38}$$

But the validity of (2.37) with (2.38) is in contradiction to Theorem 2.18. This shows that necessarily (2.28) holds, a fact which completes the proof of Theorem 2.7. □

The following Lemma will be used in the proof of Theorem 2.8.

Lemma 2.19. *Let $j : (-\infty, 0) \to \mathbb{R}$ be a function of class C^1 with $j(x) \to \infty$ as $x \nearrow 0$, and for which there exists $\delta < 0$ such that j is convex on $[\delta, 0)$. Then for every $\alpha \in (-\infty, \delta]$ there exists $\beta \in [\delta, 0)$ such that*

$$j(x) + j'(x)(y - x) \leq j(y) \quad \text{for all } x \in [\beta, 0) \text{ and } y \in [\alpha, 0). \tag{2.39}$$

Proof of Lemma 2.19. If y, $x \in [\delta, 0)$, then (2.39) holds automatically since j is convex on $[\delta, 0)$. It suffices therefore to consider the case

$$\alpha \leq y < \delta \leq x < 0.$$

In this case (2.39) can be rewritten as

$$\frac{j(x) - j(y)}{x - y} \leq j'(x), \tag{2.40}$$

and it remains to prove that this holds for all x sufficiently small.

By Lagrange's Mean Value Theorem, for every $y < \delta \leq x$ there exists $\xi_{y,x} \in$

(y,x) such that

$$\frac{j(x)-j(y)}{x-y} = j'(\xi_{y,x}). \tag{2.41}$$

Since $j(z) \to \infty$ as $z \nearrow 0$, it is possible to choose β sufficiently small so that, for all $x \in [\beta, 0)$,

$$j(x) > \sup\{j(z) : z \in [\alpha,\delta]\} + \alpha \sup\{j'(z) : z \in [\alpha,\delta]\} \tag{2.42}$$

Then, for $y \in [\alpha, \delta)$ and $x \in [\beta, 0)$,

$$\frac{j(x)-j(y)}{x-y} > \sup\{j'(z) : z \in [\alpha,\delta]\},$$

and therefore in (2.41) one must necessarily have that $\xi_{y,x} \in (\delta, x)$. But $z \mapsto j'(z)$ is increasing on $(\delta, 0)$ since j is convex there, and hence $j'(\xi_{y,x}) \leq j'(x)$. Together with (2.41), this proves (2.40). This completes the proof of the Lemma. □

Proof of Theorem 2.8. Let δ be as in the statement of the Theorem. Applying Lemma 2.19 with $j := -\log h$ and $\alpha := \min \mathcal{R}(w)$ yields $\beta \in [\delta, 0)$ such that, for all t with $w(t) \in [\beta, 0)$ and for all $s \in \mathbb{R}$,

$$\log h(w(t)) + (h'(w(t)/h(w(t))(w(s) - w(t)) - \log h(w(s)) \geq 0. \tag{2.43}$$

Fix t_1, $t_2 \in \mathcal{N}$ with $(t_1, t_2) \subset \mathbb{R} \setminus \mathcal{N}$ such that $w(t) \in [\beta, 0]$ for all $t \in (t_1, t_2)$. It follows from (2.19) and (2.43) that

$$\vartheta'(t) - \frac{h'(w(t))}{h(w(t))}\left(\frac{\cos\vartheta(t)}{h(w(t))} - 1\right) \geq 0 \quad \text{for all } t \in (t_1, t_2). \tag{2.44}$$

An examination of the proofs given in [27] of Lemma 2.16 and Lemma 2.17 reveals that their conclusion continues to hold on a particular interval (a, b) if the assumption that h satisfies (2.6) is replaced by the weaker requirements that

$$\vartheta'(t) - \frac{h'(w(t))}{h(w(t))}\left(\frac{\cos\vartheta(t)}{h(w(t))} - 1\right) \geq 0 \quad \text{for all } t \in (a, b),$$

and

$$h'(w(t)) \leq 0 \quad \text{for all } t \in (a, b).$$

This observation, combined with the fact that (2.44) holds, enables us to use the same arguments as in the proof of Theorem 2.7 to get to the required result. $\square$

2.3 Symmetry

Consider a free boundary of the form $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous, with minimal period 2π and exactly one local maximum per period. There is no loss of generality in assuming that $X = \pm\pi$ are consecutive minimizers of η, since this can always be achieved by a horizontal translation. Let X_c be a maximizer of η, where $-\pi < X_c < \pi$.

Theorem 2.20. *Suppose that $h \in C^{1,\alpha}_{\mathrm{loc}}(-\infty, 0)$ for some $\alpha \in (0,1)$. Let $(\mathcal{S}, \psi)$ be a solution of* (1.1)*, with $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where η is continuous and 2π-periodic, strictly increasing on $[-\pi, X_c]$ and strictly decreasing on $[X_c, \pi]$, for some X_c with $-\pi < X_c < \pi$. Then $X_c = 0$, $\eta(-X) = \eta(X)$ for all $X \in \mathbb{R}$, and $\psi(-X, Y) = \psi(X, Y)$ for all $(X, Y) \in \Omega$.*

Proof of Theorem 2.20. Theorem 2.2 shows that $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$ is a C^2 curve, $\psi \in C^2(\Omega \cup (\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}))$, and (2.2), (2.3) hold. Let $Y_t := \eta(\pm\pi)$ and $Y_c := \eta(X_c)$. Note that $\mathcal{S}_{\mathcal{N}} \neq \emptyset$ if and only if $Y_c = 0$, in which case $\mathcal{S}_{\mathcal{N}} = \{(X, Y_c) : X = X_c + 2k\pi,\ k \in \mathbb{Z}\}$. All the subsequent arguments in this proof apply irrespective of whether this is the case or not.

Since the harmonic conjugate φ of $-\psi$ in Ω is determined only up to an additive constant, it can be assumed that ω maps the points $\pm\pi + iY_t$ onto the points $\pm\pi + i0$, and hence $w(\pm\pi) = Y_t$. Since $\pm\pi$ are global minima for w, it follows that $w'(\pm\pi) = 0$, and therefore, by (2.1), $1 + \mathcal{C}w'(\pm\pi) \neq 0$. This implies that, on sufficiently small neighbourhoods of $X = \pm\pi$, η is a C^2 function. For every $A, B \in \mathbb{R}$ with $A < B$, let

$$\Omega_{A,B} := \{(X, Y) \in \Omega : A < X < B\}.$$

Note that it is enough to show that $X_c = 0$ and that η is an even function. Then the fact that $\psi(-X, Y) = \psi(X, Y)$ for all $(X, Y) \in \Omega$ is immediate. Indeed, if $\eta(-X) = \eta(X)$ for all $X \in \mathbb{R}$, then consider the function

$$\xi(X, Y) := \psi(X, Y) - \psi(-X, Y),$$

which is harmonic in $\Omega_{-\pi,\pi}$ and continuous on its closure. Notice that $\xi = 0$ on the finite boundary of $\Omega_{-\pi,\pi}$ and, since

$$\xi(X,Y) = \int_{-X}^{X} \psi_X(T,Y)\,dT \quad \text{for all } (X,Y) \in \Omega_{-\pi,\pi}$$

and (1.1h) holds, then $\xi(X,Y) \to 0$ as $Y \to -\infty$ within this domain, so it follows from the Maximum Principle that $\xi \equiv 0$; hence the desired conclusion.

We now prove that $X_c = 0$, and $\eta(-X) = \eta(X)$ for all $X \in \mathbb{R}$. Since each of the restrictions of η to $[-\pi, X_c]$ and $[X_c, \pi]$ is a continuous bijection onto $[Y_t, Y_c]$, let g_l and g_r denote their inverses, and consider the continuous function $g : [Y_t, Y_c] \to \mathbb{R}$ given by $g := (g_l + g_r)/2$. Note that $g(Y_t) = 0$ and $g(Y_c) = X_c$, and that symmetry of $\mathcal{S}$ is equivalent to $g \equiv 0$ on $[Y_t, Y_c]$. Let

$$\min(g) := \min\{g(Y) : Y \in [Y_t, Y_c]\}, \; \max(g) := \max\{g(Y) : Y \in [Y_t, Y_c]\}.$$

We say that a free boundary $\mathcal{S}$ has property (L) if $\min(g) \neq X_c$ and property (R) if $\max(g) \neq X_c$. Observe that, for any non-symmetric $\mathcal{S}$, at least one of the properties (L) and (R) must hold.

If $(\mathcal{S}, \psi)$ is any solution of (1.1), then it is possible to associate with it a new domain $\widetilde{\Omega}$ obtained by reflection in the vertical line $X = 0$, i.e. $\widetilde{\Omega} := \{(-X,Y) : (X,Y) \in \Omega\}$, on which a function $\tilde{\psi}$ can be defined by $\tilde{\psi}(X,Y) := \psi(-X,Y)$, for $(X,Y) \in \widetilde{\Omega}$. Denoting by $\widetilde{\mathcal{S}}$ the boundary of $\widetilde{\Omega}$, it is straightforward to confirm that $(\widetilde{\mathcal{S}}, \tilde{\psi})$ is also a solution of (1.1). With $\widetilde{X}_c$ and $\tilde{g}$ having the obvious meaning, it is very easy to check that

$$\widetilde{X}_c = -X_c \text{ and } \tilde{g} = -g.$$

If $\mathcal{S}$ does not satisfy (L), then it necessarily satisfies (R), and since

$$\min(\tilde{g}) = \min(-g) = -\max(g) \neq -X_c = \widetilde{X}_c,$$

it follows that (L) holds for $\widetilde{\mathcal{S}}$.

Hence, if non-symmetric free boundaries exist, then there exists a non-symmetric free boundary satisfying (L). We now show that this is not possible.

Assume, for definiteness of notation, that $(\mathcal{S}, \psi)$ is a solution of (1.1) such that

(L) holds for $\mathcal{S}$. Then exactly one of the following two situations must occur:

(L_1) $\min(g) \neq X_c$ and $\min(g) < 0$;

(L_2) $\min(g) \neq X_c$ and $\min(g) = 0$.

Suppose that (L_1) occurs. Let $X_0 := \min(g)$, where $-\pi < X_0 < 0$ and $X_0 < X_c$, and let Y_0 be such that $g(Y_0) = X_0$, where $Y_t < Y_0 < Y_c$. Let $X_l := g_l(Y_0)$ and $X_r := g_r(Y_0)$, so that $X_l < X_c < X_r$ and $\eta(X_l) = \eta(X_r) = Y_0$. Let $\Omega_{-\pi,X_0;X_0}$ denote the reflection of $\Omega_{-\pi,X_0}$ in the line $X = X_0$ by means of the mapping $(X,Y) \mapsto (2X_0 - X, Y)$. Note that $\Omega_{-\pi,X_0}$ can be written as a union of open horizontal line segments. It is clear from the definition of X_0 that the reflection of each of these segments is contained in Ω, hence $\Omega_{-\pi,X_0;X_0}$ is a subset of Ω. Let $X_1 := 2X_0 + \pi$, so that $X_0 < X_1 < \pi$, and let $X_2 := (X_1 + \pi)/2$. Since η is strictly decreasing on the interval (X_1, π), the domain $\Omega_{X_2,\pi;X_2}$ obtained by reflecting $\Omega_{X_2,\pi}$ in the line $X = X_2$ by means of the mapping $(X,Y) \mapsto (2X_2 - X, Y)$ is a subset of Ω. Let

$$\widetilde{\Omega} := \Omega_{-\pi,X_0;X_0} \cup \{(X_1, Y) : -\infty < Y < Y_t\} \cup \Omega_{X_2,\pi;X_2}$$

so that $\widetilde{\Omega} = \{(X,Y) : X_0 < X < X_2,\ -\infty < Y < \tilde{\eta}(X)\}$, where $\tilde{\eta}$ is a continuous function on the interval (X_0, X_2), given by

$$\tilde{\eta}(X) := \begin{cases} \eta(2X_0 - X), & \text{for } X_0 < X \leq X_1, \\ \eta(2X_2 - X), & \text{for } X_1 < X < X_2. \end{cases}$$

Moreover, if $\widetilde{\mathcal{S}} := \{(X, \tilde{\eta}(X)) : X_0 < X < X_2\}$, then $\widetilde{\mathcal{S}}$ is a C^2 curve. Since $\widetilde{\Omega}$ is included in Ω, it follows that $\tilde{\eta}(X) \leq \eta(X)$ for all $X \in (X_0, X_2)$. Together with the fact that $\tilde{\eta}(X_r) = \eta(X_r)$, this ensures that the curves $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ have the same tangent at (X_r, Y_0). Consider now the function ξ defined in $\widetilde{\Omega}$ by

$$\xi(X,Y) := \begin{cases} \psi(X,Y) - \psi(2X_0 - X, Y), & X_0 < X \leq X_1,\ -\infty < Y < \tilde{\eta}(X), \\ \psi(X,Y) - \psi(2X_2 - X, Y), & X_1 < X < X_2,\ -\infty < Y < \tilde{\eta}(X). \end{cases}$$

Since the harmonic function ψ is 2π-periodic in X, it follows that ξ is smooth across the line $X = X_1$, and harmonic in $\widetilde{\Omega}$. In addition, ξ is continuous on the closure of $\widetilde{\Omega}$, $\xi = 0$ on the lateral boundaries of this domain and, from (1.1h),

$\xi(X,Y) \to 0$ as $Y \to -\infty$ within $\widetilde{\Omega}$. Moreover, since $\psi = 0$ on $\mathcal{S}$ and $\psi < 0$ in Ω, it follows that $\xi \le 0$ on $\widetilde{\mathcal{S}}$ and $\xi < 0$ at the point $(X_c, \tilde{\eta}(X_c))$ on $\widetilde{\mathcal{S}}$. Thus, the Maximum Principle shows that $\xi < 0$ in $\widetilde{\Omega}$. Since $\xi = 0$ at (X_r, Y_0) and $\widetilde{\Omega}$ satisfies the interior sphere condition at this point (since $\widetilde{\mathcal{S}}$ is a C^2 curve), it follows from Hopf Boundary-Point Lemma that $\partial\xi/\partial n(X_r, Y_0) > 0$, where n is the outward normal to $\widetilde{\mathcal{S}}$. On the other hand, it is clear from the definition of $\widetilde{\mathcal{S}}$, the tangency of $\widetilde{\mathcal{S}}$ to $\mathcal{S}$ at (X_r, Y_0), and using also (2.3), that

$$\frac{\partial \xi}{\partial n}(X_r, Y_0) = \frac{\partial \psi}{\partial n}(X_r, Y_0) - \frac{\partial \psi}{\partial n}(X_l, Y_0) = 0.$$

The contradiction obtained shows that (L_1) cannot occur.

Suppose now that (L_2) occurs. Since $\min(g) = 0 < X_c$, it follows that the domain $\widetilde{\Omega} := \Omega_{-\pi,0;0}$ obtained by reflecting $\Omega_{-\pi,0}$ in the line $X = 0$ is a subset of Ω. Clearly $\widetilde{\Omega} = \{(X,Y) : 0 < X < \pi,\ -\infty < Y < \tilde{\eta}(X)\}$, where $\tilde{\eta}(X) := \eta(-X)$, for all $0 < X < \pi$. Let ξ be defined in $\widetilde{\Omega}$ by

$$\xi(X,Y) := \psi(X,Y) - \psi(-X,Y).$$

Then ξ is harmonic in $\widetilde{\Omega}$ and continuous on its closure. In addition, $\xi = 0$ on the lateral boundaries of this domain, and $\xi(X,Y) \to 0$ as $Y \to -\infty$ within $\widetilde{\Omega}$. Moreover, since $\psi = 0$ on $\mathcal{S}$ and $\psi < 0$ in Ω, it follows that $\xi \le 0$ on $\widetilde{\mathcal{S}}$ and $\xi < 0$ at the point $(X_c, \tilde{\eta}(X_c)$ on $\widetilde{\mathcal{S}}$. Thus, the Maximum Principle shows that $\xi < 0$ in $\widetilde{\Omega}$. Note also that $\xi(\pi, Y_t) = 0$. Let us now calculate the first and second order partial derivatives of ξ at (π, Y_t). Since η is a C^2 function for X close to $\pm\pi$, and ψ can be extended as a C^2 function in a neighbourhood of $(\pm\pi, Y_t)$, the following calculations are valid. Differentiating with respect to X the relation $\psi(X, \eta(X)) = 0$ at $X = \pm\pi$ leads to

$$\psi_X + \psi_Y \eta' = 0,$$

and since $\eta'(\pm\pi) = 0$, it follows that $\psi_X(\pm\pi, Y_t) = 0$, and hence $\xi_X(\pi, Y_t) = 0$. The periodicity of ψ shows that $\xi_Y(\pi, Y_t) = \xi_{XX}(\pi, Y_t) = \xi_{YY}(\pi, Y_t) = 0$. Differentiating now with respect to X the relation $|\nabla\psi|^2(X, \eta(X)) - \lambda(\eta(X)) = 0$

at $X = \pm\pi$ leads to

$$2\psi_X(\psi_{XX} + \psi_{XY}\eta') + 2\psi_Y(\psi_{XY} + \psi_{YY}\eta') - \lambda'(\eta)\eta' = 0,$$

and, since $\eta'(\pm\pi) = 0$, $\psi_X(\pm\pi, Y_t) = 0$, and $\psi_Y(\pm\pi, Y_t) = h(Y_t) \neq 0$, it follows that $\psi_{XY}(\pm\pi, Y_t) = 0$, and hence $\xi_{XY}(\pi, Y_t) = 0$. Thus, all first and second order partial derivatives of ξ vanish at the point (π, Y_t). But this is in contradiction with the following version of Serrin Corner-Point Lemma [24], applied to $Q := (\pi, Y_t)$, $\Theta := \widetilde{\Omega} \cup \{(\pi, Y) : -\infty < Y < Y_t\} \cup \Omega_{\pi,2\pi}$ and $\Theta_0 := \widetilde{\Omega}$.

Lemma 2.21. *Let Θ be a domain in $\mathbb{R}^2$, Q a point on $\partial\Theta$ and assume that, locally in neighbourhood of Q, the boundary of Θ is a C^2 curve. Let Θ_0 denote the portion of Θ lying on some particular side of the line containing the normal to $\partial\Theta$ at Q. Suppose that $\xi \in C^2(\overline{\Theta_0})$ is harmonic in Θ_0, with $\xi \leq 0$ in Θ_0 and $\xi(Q) = 0$. If m is any direction which enters Θ_0 non-tangentially, then either $\partial\xi/\partial m < 0$, or $\partial^2\xi/\partial m^2 < 0$, unless $\xi \equiv 0$ in $\overline{\Theta_0}$.*

This shows that (L_2) cannot occur either. In conclusion, no non-symmetric free boundaries exist, and the proof of Theorem 2.20 is completed. □

Chapter 3

Behaviour of Free Boundaries at Singular Points

In this chapter we study the existence of lateral tangents to free boundaries at isolated stagnation points. We show that, for a large class of Bernoulli problems, a free boundary which is symmetric with respect to a vertical line through a singular point must necessarily have a corner at that point, and we give a formula for the contained angle. This result extends the first Stokes conjecture in the theory of hydrodynamic waves, which was originally proved by Amick, Fraenkel and Toland [1] and Plotnikov [21]. By means of an example we show that free boundaries need not always have a corner at a singular point. We also show that, for a large class of problems, if a free boundary has a corner at an asymmetric stagnation point, then the lateral tangents must be symmetric with respect to the vertical line passing through that point, and enclose an angle of the expected size. The proofs are based on hard analytic estimates for nonlinear pseudo-differential and integral operator equations.

3.1 Estimates on Singular Integral Operators

Many of the equations studied in the thesis involve singular integral operators associated to the kernels

$$K(t,s) = \frac{1}{\pi}\log\left|\frac{\sin\frac{1}{2}(t+s)}{\sin\frac{1}{2}(t-s)}\right|, \quad t,s \in (0,\pi], \tag{3.1}$$

and

$$k(x,y) = \frac{1}{\pi}\log\frac{x+y}{|x-y|}, \qquad x,y \in (0,\infty). \tag{3.2}$$

It is natural to start by collecting some estimates on these operators. These estimates will be extensively used in this chapter and the next.

An equivalent expression for K is

$$K(t,s) = \frac{1}{\pi}\log\left|\frac{\tan\frac{t}{2}+\tan\frac{s}{2}}{\tan\frac{t}{2}-\tan\frac{s}{2}}\right| \qquad \text{for all } t,s \in (0,\pi). \tag{3.3}$$

We will need the following explicit evaluations of integrals:

$$\int_0^1 \log\frac{1+u}{|1-u|}\frac{1}{u}\,du = \int_1^\infty \log\frac{1+u}{|1-u|}\frac{1}{u}\,du = \frac{\pi^2}{4}. \tag{3.4}$$

Indeed, the equality of the integrals in (3.4) can be seen by making the change of variables $u \leftrightarrow u^{-1}$. To evaluate the first integral, we expand the integrand into a power series, for $u \in (0,1)$,

$$\log(1+u) = \sum_{m=1}^\infty (-1)^{m-1}\frac{u^m}{m}, \qquad \log(1-u) = -\sum_{m=1}^\infty \frac{u^m}{m}.$$

This shows that, for $u \in (0,1)$,

$$\frac{1}{u}\log\frac{1+u}{1-u} = 2\sum_{k=0}^\infty \frac{u^{2k}}{2k+1} \tag{3.5}$$

and hence

$$\int_0^1 \log\frac{1+u}{1-u}\frac{1}{u}\,du = 2\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{\pi^2}{4},$$

where we have used the known fact that $\sum_{m=1}^\infty 1/m^2 = \pi^2/6$.

It follows that, for any $x \in (0,\infty)$,

$$\int_0^\infty k(x,y)\frac{1}{y}\,dy = \frac{1}{\pi}\int_0^\infty \log\frac{1+u}{|1-u|}\frac{1}{u}\,du = \frac{\pi}{2}, \tag{3.6}$$

where we have used the change of variables $y = xu$ and the homogeneity of k.

Another consequence of (3.5) is that

$$\log(1+u) - \log(1-u) \geq 2u \quad \text{for all } u \in (0,1). \tag{3.7}$$

It is also easy to see that

$$k(1,u) \geq \frac{\log 3}{\pi} \text{ for all } u \in [1/2, 2]. \tag{3.8}$$

Note also that

$$K(t,s) \leq k(t,s) \quad \text{for all } t,s \in (0,\pi],\ t \neq s, \tag{3.9}$$

since this is equivalent to

$$\frac{\sin(\frac{1}{2}(t+s))}{t+s} \leq \frac{\sin(\frac{1}{2}|t-s|)}{|t-s|},$$

which is true because the mapping $u \mapsto (\sin u)/u$ is decreasing on $(0,\pi]$.

For each $\alpha \in (0,1)$, let $\bar{u}_\alpha$ be the unique root of the equation $u = \alpha \tan u$ in the interval $(0,\pi/2)$. The function $g_\alpha : (0,\bar{u}_\alpha] \to \mathbb{R}$ given by $g_\alpha(u) := (\sin u)/u^\alpha$, is increasing on $(0,\bar{u}_\alpha]$, since

$$g'_\alpha(u) = \frac{\cos u}{u^{\alpha+1}}\Big(u - \alpha \tan u\Big) \geq 0 \quad \text{on } (0,\bar{u}_\alpha].$$

Hence

$$\frac{\sin(\frac{1}{2}|t-s|)}{|t-s|^\alpha} \leq \frac{\sin(\frac{1}{2}(t+s))}{(t+s)^\alpha} \quad \text{for all } t,s \in (0,\bar{u}_\alpha],\ t \neq s,$$

so that

$$\alpha\, k(t,s) \leq K(t,s) \quad \text{for all } t,s \in (0,\bar{u}_\alpha],\ t \neq s.$$

In particular, if $\alpha := \pi/4$, it follows that

$$\frac{\pi}{4}\, k(t,s) \leq K(t,s) \quad \text{for all } t,s \in (0,\pi/4],\ t \neq s. \tag{3.10}$$

Lemma 3.1. *Let $\varpi : [1,\infty) \to [0,\infty)$ be given by*

$$\varpi(\gamma) := \int_0^\infty |k(\gamma,z) - k(1,z)| \frac{1}{z}\, dz \quad \textit{for all } \gamma \geq 1. \tag{3.11}$$

Then $\varpi(\gamma) = 4\int_1^{\sqrt{\gamma}} k(1,z)\frac{1}{z}\,dz$, *and therefore* $\lim_{\gamma\to 1^+}\varpi(\gamma) = 0$.

Proof of Lemma 3.1. For $\gamma > 1$, we claim that the set of values of $z \in (0,\infty)$ for which $k(\gamma,z) \geq k(1,z)$ coincides with the interval $[\sqrt{\gamma},\infty)$. Indeed, the claim is proved by the following chain of equivalences:

$$\begin{aligned} & k(\gamma,z) \geq k(1,z) \Leftrightarrow (\gamma+z)^2(1-z)^2 \geq (\gamma-z)^2(1+z)^2 \\ \Leftrightarrow\ & (\gamma^2+z^2+2\gamma z)(1+z^2-2z) \geq (\gamma^2+z^2-2\gamma z)(1+z^2+2z) \\ \Leftrightarrow\ & 2\gamma z(1+z^2) - 2z(\gamma^2+z^2) \geq -2\gamma z(1+z^2) + 2z(\gamma^2+z^2) \\ \Leftrightarrow\ & \gamma(1+z^2) \geq \gamma^2+z^2 \Leftrightarrow (z^2-\gamma)(\gamma-1) \geq 0 \Leftrightarrow z^2 \geq \gamma. \end{aligned}$$

Hence

$$\begin{aligned} \varpi(\gamma) &= \left(\int_0^{\sqrt{\gamma}} k(1,z)\frac{dz}{z} - \int_0^{\sqrt{\gamma}} k(\gamma,z)\frac{dz}{z}\right) + \left(\int_{\sqrt{\gamma}}^{\infty} k(\gamma,z)\frac{dz}{z} - \int_{\sqrt{\gamma}}^{\infty} k(1,z)\frac{dz}{z}\right) \\ &= \int_0^{\sqrt{\gamma}} k(1,z)\frac{dz}{z} - \int_0^{1/\sqrt{\gamma}} k(1,u)\frac{du}{u} + \int_{1/\sqrt{\gamma}}^{\infty} k(1,u)\frac{du}{u} - \int_{\sqrt{\gamma}}^{\infty} k(1,z)\frac{dz}{z} \\ &= 2\int_{1/\sqrt{\gamma}}^{\sqrt{\gamma}} k(1,z)\frac{1}{z}\,dz = 4\int_1^{\sqrt{\gamma}} k(1,z)\frac{1}{z}\,dz, \end{aligned}$$

where the change of variables formula has been used several times in obvious ways. It is now immediate that $\lim_{\gamma\to 1^+}\varpi(\gamma) = 0$. □

Lemma 3.2. *Let* $\varrho \in L^\infty(0,\infty)$ *and let* $\varsigma : (0,\infty) \to \mathbb{R}$ *be defined, for all* $x \in (0,\infty)$, *by*

$$\varsigma(x) = \int_0^\infty k(x,y)\frac{1}{y}\varrho(y)\,dy.$$

Then $\varsigma \in C_b(0,\infty)$ *and, for all* $x_1, x_2 \in (0,\infty)$ *with* $x_1 < x_2$,

$$|\varsigma(x_2) - \varsigma(x_1)| \leq \varpi(x_2/x_1)\,||\varrho||_{L^\infty(0,\infty)}, \tag{3.12}$$

where ϖ *is given by* (3.11).

Proof of Lemma 3.2. Note first that ς is a bounded function since, by (3.6),

$$|\varsigma(x)| \leq \int_0^\infty k(x,y)\frac{1}{y}\,dy\;||\varrho||_{L^\infty(0,\infty)} = \frac{\pi}{2}||\varrho||_{L^\infty(0,\infty)}, \tag{3.13}$$

for every $x \in (0, \infty)$. Let $x_1, x_2 \in (0, \infty)$ with $x_1 < x_2$. It follows by the change of variables $y = x_1 z$ that

$$\begin{aligned} |\varsigma(x_2) - \varsigma(x_1)| &\leq \int_0^\infty |k(x_2, y) - k(x_1, y)| \frac{1}{y} \, dy \; ||\varrho||_{L^\infty(0,\infty)} \\ &= \int_0^\infty |k(x_2/x_1, z) - k(1, z)| \frac{1}{z} \, dz \; ||\varrho||_{L^\infty(0,\infty)}, \end{aligned}$$

which proves (3.12). The continuity of ς is immediate from Lemma 3.1. □

Corollary 3.3. *Let $\{\varrho_n\}_{n\geq 1}$ be a uniformly bounded sequence of functions on $(0, \infty)$, and let $\varsigma_n : (0, \infty) \to \mathbb{R}$ be defined, for all $x \in (0, \infty)$, by*

$$\varsigma_n(x) = \int_0^\infty k(x, y) \frac{1}{y} \varrho_n(y) \, dy.$$

Then the sequence of functions $\{\varsigma_n\}_{n\geq 1}$ is uniformly bounded and equicontinuous on $(0, \infty)$.

Proof of Corollary 3.3. The uniform boundedness of $\{\varsigma_n\}_{n\geq 1}$, is clear from (3.13). The equicontinuity of the sequence $\{\varsigma_n\}_{n\geq 1}$ follows from (3.12) and Lemma 3.1. □

Corollary 3.4. *Under the assumptions of Lemma 3.2, the following holds: for every sequence $\{\epsilon_n\}_{n\geq 1}$ with $\epsilon_n > 0$ for all n, the sequence of functions $\{\varsigma_n\}_{n\geq 1}$, defined by $\varsigma_n(x) := \varsigma(\epsilon_n x)$ for all $x \in (0, \infty)$, is uniformly bounded and equicontinuous on $(0, \infty)$.*

Proof of Corollary 3.4. This can be seen either directly from (3.13) and (3.12), or as a particular case of Corollary 3.3 since, for all $n \geq 1$ and $x \in (0, \infty)$,

$$\varsigma_n(x) = \int_0^\infty k(x, y) \frac{1}{y} \varrho_n(y) \, dy,$$

where $\varrho_n(y) := \varrho(\epsilon_n x)$, for all $y \in (0, \infty)$. □

Lemma 3.5. *Under the assumptions of Lemma 3.2, suppose in addition that $\lim_{y\to 0^+} \varrho(y) = a$. Then $\lim_{x\to 0^+} \varsigma(x) = a\pi/2$.*

Proof of Lemma 3.5. By Lemma 3.2, ς is a bounded function. Fix $\varepsilon > 0$. Let

$b \in (0, \infty)$ be such that

$$a - \varepsilon \leq \varrho(y) \leq a + \varepsilon \quad \text{for all } y \in (0, b).$$

Then one can write, for $x \in (0, \infty)$,

$$\varsigma(x) = \int_0^b k(x,y) \frac{1}{y} \varrho(y)\, dy + \int_b^\infty k(x,y) \frac{1}{y} \varrho(y)\, dy =: P(x) + Q(x). \tag{3.14}$$

Clearly, for $x \in (0, \infty)$,

$$(a - \varepsilon) \int_0^b k(x,y) \frac{1}{y}\, dy \leq P(x) \leq (a + \varepsilon) \int_0^b k(x,y) \frac{1}{y}\, dy,$$

so that, using the change of variables $y = xu$,

$$(a - \varepsilon) \int_0^{b/x} k(1,u) \frac{1}{u}\, du \leq P(x) \leq (a + \varepsilon) \int_0^{b/x} k(1,u) \frac{1}{u}\, du.$$

As $x \to 0^+$, it follows using (3.6) that

$$(a - \varepsilon) \frac{\pi}{2} \leq \liminf_{x \to 0+} P(x) \leq \limsup_{x \to 0+} P(x) \leq (a + \varepsilon) \frac{\pi}{2}. \tag{3.15}$$

Note also that, for $x \in (0, \infty)$,

$$|Q(x)| \leq \int_b^\infty k(x,y) \frac{1}{y}\, dy ||\varrho||_{L^\infty(0,\infty)} = \int_{b/x}^\infty k(1,u) \frac{1}{u}\, du ||\varrho||_{L^\infty(0,\infty)},$$

so that

$$\lim_{x \to 0+} Q(x) = 0. \tag{3.16}$$

Using (3.15) and (3.16) in (3.14) yields

$$(a - \varepsilon) \frac{\pi}{2} \leq \liminf_{x \to 0+} \varsigma(x) \leq \limsup_{x \to 0+} \varsigma(x) \leq (a + \varepsilon) \frac{\pi}{2}. \tag{3.17}$$

Since (3.17) holds for all $\varepsilon > 0$, it follows that $\lim_{x \to 0+} \varsigma(x) = a\pi/2$, as required.

□

The following lemma deals with a situation which will be frequently encountered in this and the next Chapter. The result is essentially [15, Lemma 1], (see also [2, Lemma 10.4.3]), with a slightly simplified proof.

Lemma 3.6. *Let $\kappa > 0$. Let $I := (0, \infty)$ or $I := (0, a]$ for some $a > 0$, and let $\hat{I} := \{2x : x \in I\}$. Let $\phi : \hat{I} \to \mathbb{R}$ be a continuous function with $0 \le \phi \le \pi/2$ on $\hat{I}$, such that, for all $x \in I$,*

$$\phi(x) \ge \kappa \int_0^{2x} k(x, y) \frac{\sin \phi(y)}{\nu + \int_0^y \sin \phi(u)\, du}\, dy, \tag{3.18}$$

where $\nu \ge 0$. Then there exists $m_1 > 0$, independent of I, ν and ϕ, such that

$$\nu + \int_0^x \sin \phi(u)\, du \ge m_1 x \quad \text{for all } x \in I. \tag{3.19}$$

Moreover, if $\nu = 0$, then there exists $m_2 > 0$, independent of I and ϕ, such that $\phi(x) \ge m_2$ for all $x \in I$.

Proof. Fix $x \in I$. For every $z \in [x, 2x]$, it follows from (3.18) that

$$\begin{aligned} \phi(z) &\ge \kappa \int_x^{2x} k(z, y) \frac{\sin \phi(y)}{\nu + \int_0^y \sin \phi(u)\, du}\, dy \\ &\ge \frac{\kappa \log 3}{\pi} \Big\{ \log \Big(\nu + \int_0^{2x} \sin \phi(u)\, du \Big) - \log \Big(\nu + \int_0^{x} \sin \phi(u)\, du \Big) \Big\}, \end{aligned} \tag{3.20}$$

where we have used the fact that, for $y, z \in [x, 2x]$, one has $z/y \in [1/2, 2]$, and hence $k(y, z) = k(1, z/y) \ge \log 3/\pi$, by (3.8). Integrating (3.20) yields

$$\frac{1}{x} \int_x^{2x} \phi(z)\, dz \ge \frac{\kappa \log 3}{\pi} \log \left\{ 1 + \frac{\int_x^{2x} \sin \phi(u)\, du}{\nu + \int_0^x \sin \phi(u)\, du} \right\}. \tag{3.21}$$

Since $0 \le \phi \le \pi/2$ implies that

$$\frac{2}{\pi} \phi \le \sin \phi \le \phi,$$

it follows from (3.21) that

$$\frac{\pi^2}{2\kappa \log 3} \frac{1}{x} \int_x^{2x} \sin \phi(u)\, du \ge \log \left\{ 1 + \frac{\int_x^{2x} \sin \phi(u)\, du}{\nu + \int_0^x \sin \phi(u)\, du} \right\}. \tag{3.22}$$

Since the left-hand side is bounded by $c := \pi^2/(\kappa \log 3)$, it follows that

$$\frac{\int_x^{2x} \sin\phi(u)\, du}{\nu + \int_0^x \sin\phi(u)\, du}$$

is bounded, by M say, independently of x, ϕ and ν. Since there exists $C > 0$ such that $\log(1+N) \geq CN$ for all $N \in [0, M]$, it follows from (3.22) that

$$\frac{1}{x}\int_x^{2x} \sin\phi(u)\, du \geq \frac{C}{c}\,\frac{\int_x^{2x} \sin\phi(u)\, du}{\nu + \int_0^x \sin\phi(u)\, du},$$

and hence (3.19) holds with $m_1 := C/c$. This is true for all $x \in I$, and m_1 is independent of I, ϕ and ν, as required.

Suppose now that $\nu = 0$. It follows from (3.18) that one can write, for all $x \in I$,

$$\phi(x) \geq \kappa \int_0^x k(x,y) \frac{\sin\phi(y)}{\int_0^y \sin\phi(u)\, du}\, dy,$$

and hence, since $\int_0^y \sin\phi(u)\, du \leq y$ for all $y \in (0, x)$,

$$\phi(x) \geq \kappa \int_0^x k(x,y) \frac{1}{y} \sin\phi(y)\, dy.$$

Using now (3.7) and (3.19) yields

$$\phi(x) \geq \frac{2\kappa}{\pi}\,\frac{1}{x}\int_0^x \sin\phi(y)\, dy \geq \frac{2\kappa m_1}{\pi} \quad \text{for all } x \in I,$$

and this completes the proof of Lemma 3.6. □

3.2 Symmetric Singular Free Boundaries

Let $(\mathcal{S}, \psi)$ be a solution of (1.1) and consider an isolated symmetric stagnation point. There is no loss of generality in assuming that it is located at $(0,0)$, since this can be achieved by horizontal translation. Under a weak assumption on the nonlinearity h, we show that $\mathcal{S}$ must have a corner at that point, and we give a formula for the contained angle.

For h as in (1.4) and whose restriction to $(-\delta, 0)$ is of class C^1 for some

$\delta > 0$, let f be given by (1.40) and F as in (1.41), and consider the function $E : (0,\delta) \to \mathbb{R}$ given by

$$E(r) = \frac{f'(r)F(r)}{f^2(r)} \quad \text{for all} \quad r \in (0,\delta). \tag{3.23}$$

Theorem 3.7. *Let h be as in (1.4) and such that, for some $\delta > 0$, the restriction of h to $(-\delta, 0)$ is of class C^1. With E as in (3.23), suppose that*

$$\text{there exists } \mu \in (0,1) \text{ such that } \lim_{r\to 0^+} E(r) = \mu. \tag{3.24}$$

Let $(\mathcal{S}, \psi)$ be a solution of (1.1), where $\mathcal{S}$ contains the point $(0,0)$ and is symmetric with respect to the vertical line $X = 0$. In addition, suppose that there exists $X_0 \in (0,\pi]$, $Y_0 < 0$ and $\eta : [-X_0, X_0] \to \mathbb{R}$ a continuous even function, strictly decreasing on $[0, X_0]$, and such that, if $\mathcal{R}$ denotes the rectangle $\{(X,Y) : -X_0 \leq X \leq X_0, Y_0 \leq Y \leq 0\}$, then

$$\mathcal{S} \cap \mathcal{R} = \{(X, \eta(X)) : -X_0 \leq X \leq X_0\}.$$

Then η is continuously differentiable on $[-X_1, 0) \cup (0, X_1]$ for some $X_1 \in (0, X_0]$, and $\lim_{X\to\pm 0} \eta'(X) = \mp \tan(\mu\pi/2)$, where μ is given by (3.24).

We start by deriving, from the geometric properties of $\mathcal{S}$ in Theorem 3.7, additional analytic properties of the solution θ of (1.42) associated to $\mathcal{S}$. The largest part of this Section is devoted to studying the behaviour at a singular point for solutions of (1.42) with these additional properties. The main result is Theorem 3.9, from where the conclusion of Theorem 3.7 is immediate.

The following considerations, which will also be useful in Section 4, are relevant to a more general context than that in Theorem 3.7, since symmetry of $\mathcal{S}$ is not required.

Suppose that h is as in Theorem 3.7 and $(\mathcal{S}, \psi)$ is a solution of (1.1) such that the following holds: there exists $X_0 \in (0,\pi]$, $Y_0 < 0$ and $\eta : [-X_0, X_0] \to \mathbb{R}$ a continuous function, strictly increasing on $[-X_0, 0]$, strictly decreasing on $[0, X_0]$, with $\eta(0) = 0$, and such that

$$\mathcal{S} \cap \mathcal{R} = \{(X, \eta(X)) : -X_0 \leq X \leq X_0\},$$

where $\mathcal{R}$ denotes the rectangle $\{(X, Y) : -X_0 \leq X \leq X_0, Y_0 \leq Y \leq 0\}$. We may assume that φ, a harmonic conjugate of $-\psi$ satisfies $\varphi(0,0) = 0$. With w as in Chapter 1, one obtains that $\mathcal{S}$ can be parametrized by (1.23), with the origin in the (X, Y)-plane corresponding to $t = 0$, a stagnation point. Moreover, ϑ satisfies

$$\vartheta = -\mathcal{C}\tau, \tag{3.25a}$$

$$\tau(s) = -\log\left\{h\left(H^{-1}\left(\int_0^s \sin\vartheta(v)\,dv\right)\right)\right\}, \quad s \in \mathbb{R}, \tag{3.25b}$$

$$[\tau] = 0, \tag{3.25c}$$

$$\left\{s \mapsto \frac{1}{h(H^{-1}(\int_0^s \sin\vartheta(v)\,dv))}\right\} \in L^1_{2\pi}. \tag{3.25d}$$

The assumptions on the regularity of h and on the geometry of $\mathcal{S}$ close to the origin imply, using (1.23) and (1.30), that there exists $t_0 \in (0, \pi]$ such that

$$\vartheta \text{ is continuous on } (-t_0, 0) \cup (0, t_0), \tag{3.25e}$$

$$\int_0^t \sin\vartheta(v)\,dv < 0 \quad \text{for all} \quad t \in (-t_0, 0) \cup (0, t_0). \tag{3.25f}$$

and

$$\begin{aligned} &\cos\vartheta(t) \geq 0 \text{ on } (-t_0, 0) \cup (0, t_0),\\ &\sin\vartheta(t) \geq 0 \text{ on } (-t_0, 0), \qquad \sin\vartheta(t) \leq 0 \text{ on } (0, t_0). \end{aligned}$$

It follows that there exist k_1, $k_2 \in \mathbb{Z}$ such that, for all $t \in (0, t_0)$,

$$\vartheta(t) \in \left[2k_1\pi - \frac{\pi}{2}, 2k_1\pi\right], \qquad \vartheta(-t) \in \left[2k_2\pi, 2k_2\pi + \frac{\pi}{2}\right].$$

We now recall for easy reference the result in [27, Lemma 3.10].

Lemma 3.8. *Suppose that for $t \in (0, t_0)$, $t_0 > 0$,*

$$m^+\pi \leq \vartheta(t) \leq M^+\pi \quad \textit{and} \quad m^-\pi \leq \vartheta(-t) \leq M^-\pi.$$

Then

$$m^+ \leq M^- \textit{ and } m^- < M^+ + 1.$$

It is now immediate from Lemma 3.8 that necessarily $k_1 = k_2 =: k$, so that, for all $t \in (0, t_0)$,

$$\vartheta(t) \in \left[2k\pi - \frac{\pi}{2}, 2k\pi\right], \qquad \vartheta(-t) \in \left[2k\pi, 2k\pi + \frac{\pi}{2}\right]. \tag{3.25g}$$

We therefore deduce that the following holds for $\theta := -\vartheta$, with $n := -k$.

$$\theta = C\tau, \tag{3.26a}$$

$$\tau(s) = -\log\left\{f\left(F^{-1}\left(\int_0^s \sin\theta(v)\,dv\right)\right)\right\}, \quad s \in \mathbb{R} \tag{3.26b}$$

$$[\tau] = 0, \tag{3.26c}$$

$$\left\{s \mapsto \frac{1}{f(F^{-1}(\int_0^s \sin\theta(v)\,dv))}\right\} \in L^1_{2\pi}, \tag{3.26d}$$

and there exists $t_0 \in (0, \pi]$ such that

$$\theta \text{ is continuous on } (-t_0, 0) \cup (0, t_0), \tag{3.26e}$$

$$\int_0^t \sin\theta(v)\,dv > 0 \quad \text{for all} \quad t \in (-t_0, 0) \cup (0, t_0), \tag{3.26f}$$

$$\theta(t) \in \left[2n\pi, 2n\pi + \frac{\pi}{2}\right], \ \theta(-t) \in \left[2n\pi - \frac{\pi}{2}, 2n\pi\right] \ \text{ for all } \ t \in (0, t_0). \tag{3.26g}$$

We now specialize the preceding considerations to the setting of Theorem 3.7. Since $\mathcal{S}$ is symmetric, a simple argument similar to that used at the beginning of the proof of Theorem 2.20 shows that ψ is necessarily an even function with respect to the X variable. We may suppose that φ, a harmonic conjugate of $-\psi$, in addition to satisfying $\varphi(0,0) = 0$, is odd in X. It follows that w is an even function, $\gamma = 0$ in (1.23), and the function θ satisfying (3.26) is odd. This implies that necessarily $n = 0$ in (3.26g). Now Theorem 3.9 below shows that $\lim_{t\to 0+}\theta(t) = \mu\pi/2$, which implies the result claimed by Theorem 3.7.

Theorem 3.9. *Let $f : [0,\infty) \to [0,\infty)$ be continuous, such that $f(0) = 0$, $f > 0$ on $(0,\infty)$, and $f \in C^1(0,\delta)$ for some $\delta > 0$. With E as in* (3.23), *suppose that* (3.24) *holds. Let θ be an odd function which satisfies* (3.26) *with $n = 0$. Then $\lim_{t\to 0+}\theta(t) = \mu\pi/2$, where μ is given by* (3.24).

We remark that the proof of Theorem 3.9 which will be given here does not use the assumption (3.26c), and only requires that $\tau \in L^1_{2\pi}$ rather than the stronger assumption (3.26d). The proof would also be valid if it were assumed in (3.24) that $\mu \in (0,1]$ rather than $\mu \in (0,1)$.

The proof of Theorem 3.9 is quite long and elaborate, and consists of several steps. We start with an outline of the strategy.

The first step is the following result, which shows that θ is necessarily discontinuous at the origin.

Proposition 3.10. *Under the assumptions of Theorem 3.9,* $\liminf_{t\to 0^+} \theta(t) > 0$.

Then, to prove that $\lim_{t\to 0^+} \theta(t) = \mu\pi/2$, we study the limit points of $\theta(t)$ as $t \to 0^+$. As in [1], our analysis involves the integral equation

$$\phi^*(x) = \mu \int_0^\infty k(x,y) \frac{\sin \phi^*(y)}{\int_0^y \sin \phi^*(u)\, du}\, dy, \qquad x \in (0,\infty), \tag{3.27a}$$

$$\inf_{x\in(0,\infty)} \phi^*(x) > 0 \quad \textit{and} \quad \sup_{x\in(0,\infty)} \phi^*(x) \le \pi/2, \tag{3.27b}$$

where k is given by (3.2). Solutions of (3.27) are required to belong to the space $C_b(0,\infty)$ of bounded continuous functions on $(0,\infty)$, to satisfy the relation (3.27a) at every point, and the bounds (3.27b). The connection between the limit points of $\theta(t)$ as $t \to 0^+$ and the solutions of (3.27) is given by the following result.

Theorem 3.11. *Under the assumptions of Theorem 3.9, suppose that there exists a sequence $\varepsilon_n \to 0^+$ such that $\theta(\varepsilon_n) \to a$, where $0 \le a \le \pi/2$. Then $a > 0$ and there exists a solution ϕ^* of (3.27), with μ given by (3.24), such that $\phi^*(1) = a$.*

It is straightforward that Theorem 3.9 now follows from Theorem 3.11 and the following theorem.

Theorem 3.12. *Let $\mu \in (0,1]$. The only solution ϕ^* of (3.27) is the constant function $\mu\pi/2$.*

We now give the proofs of Proposition 3.10, Theorem 3.11 and Theorem 3.12.

3.2.1 Preliminary Estimate - Proof of Proposition 3.10

Let $s_0 \in (0,t_0)$ be chosen sufficiently small so that $F^{-1}(\int_0^s \sin\theta(v)\, dv) < \delta$ for all $s \in (0,s_0)$. Then τ is a continuously differentiable function on $(0,s_0)$ and

$$-\tau'(s) = E\Big(F^{-1}\Big(\int_0^s \sin\theta(v)\, dv\Big)\Big) \frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\, dv} \qquad \text{for all } s \in (0,s_0).$$

It follows from (3.24) that there exists s_1 with $0 < s_1 < s_0$ such that τ is a decreasing function on $(0, s_1)$ and, moreover,

$$-\tau'(s) \geq \frac{\mu}{2} \frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\,dv} \qquad \text{for all } s \in (0, s_1). \tag{3.28}$$

Since $\tau \in L^1_{2\pi}$, the monotonicity of τ shows that $\int_0^s \tau(v)\,dv \geq s\tau(s) \geq 0$ for all $s \in (0, s_1)$, which implies that $\lim_{s\to 0+} s\tau(s) = 0$.

Since τ is an even function, the definition of a conjugate function shows that, for any s_2 with $0 < s_2 < s_1$, and for all $t \in (0, s_2)$,

$$\begin{aligned}
\theta(t) &= \frac{1}{2\pi}\int_0^\pi \Big\{\cot\Big(\frac{1}{2}(t-s)\Big) + \cot\Big(\frac{1}{2}(t+s)\Big)\Big\}\{\tau(s) - \tau(s_2)\}\,ds \\
&= \frac{1}{2\pi}\int_0^{s_2} \Big\{\cot\Big(\frac{1}{2}(s+t)\Big) - \cot\Big(\frac{1}{2}(s-t)\Big)\Big\}\{\tau(s) - \tau(s_2)\}\,ds \\
&\quad + \frac{1}{2\pi}\int_{s_2}^\pi \Big\{\cot\Big(\frac{1}{2}(s-t)\Big) - \cot\Big(\frac{1}{2}(s+t)\Big)\Big\}\{\tau(s_2) - \tau(s)\}\,ds \\
&=: P_{s_2}(t) + Q_{s_2}(t).
\end{aligned}$$

We now show that, if s_2 is suitably chosen, then $Q_{s_2}(t)$ is non-negative for all t sufficiently small. It is clear that

$$\begin{aligned}
\lim_{t\to 0+}\frac{Q_{s_2}(t)}{t} &= \lim_{t\to 0+}\frac{1}{2\pi}\int_{s_2}^\pi \frac{1}{t}\Big\{\cot\Big(\frac{1}{2}(s-t)\Big) - \cot\Big(\frac{1}{2}(s+t)\Big)\Big\}\{\tau(s_2) - \tau(s)\}\,ds \\
&= \frac{1}{2\pi}\int_{s_2}^\pi \frac{1}{\sin^2\frac{s}{2}}\{\tau(s_2) - \tau(s)\}\,ds \\
&\geq \frac{1}{2\pi}\int_{s_1}^\pi \frac{1}{\sin^2\frac{s}{2}}\{\tau(s_2) - \tau(s)\}\,ds,
\end{aligned}$$

since τ is a decreasing function on the interval (s_2, s_1). It follows that

$$\begin{aligned}
2\pi\lim_{t\to 0+}\frac{Q_{s_2}(t)}{t} &\geq \tau(s_2)\int_{s_1}^\pi \frac{1}{\sin^2\frac{s}{2}}\,ds - \int_{s_1}^\pi \frac{\tau(s)}{\sin^2\frac{s}{2}}\,ds \\
&\geq 2\tau(s_2)\cot\frac{s_1}{2} - ||\tau||_{L^1(0,\pi)}\frac{1}{\sin^2\frac{s_1}{2}}.
\end{aligned}$$

Since $\lim_{s\to 0+}\tau(s) = +\infty$, it follows that there exists $s_2 \in (0, s_1)$ such that $\tau(s_2) > (\sin s_1)^{-1}||\tau||_{L^1(0,\pi)}$. Fix s_2 with this property. It can be assumed with no loss of generality that $0 < s_2 \le \pi/4$. It is a consequence of $\lim_{t\to 0^+} Q_{s_2}(t)/t > 0$ that there exists t_1 with $0 < t_1 < s_2$ such that $Q_{s_2}(t) \ge 0$ for all $t \in (0, t_1)$.

For any $t \in (0, s_2)$, let us write $P_{s_2}(t)$ in a different form. We have that

$$P_{s_2}(t) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{\varepsilon}^{s_2} \left\{ \cot\left(\frac{1}{2}(s+t)\right) - \cot\left(\frac{1}{2}(s-t)\right)\right\}\{\tau(s) - \tau(s_2)\}\, ds. \quad (3.29)$$

Integrate by parts on the interval (ε, s_2), and then let $\varepsilon \to 0$, to conclude that

$$P_{s_2}(t) = -\int_0^{s_2} K(t,s)\tau'(s)\, ds,$$

where K is given by (3.1). In justifying the validity of this formula, one uses the fact that, for any $t \in (0, s_2)$,

$$\lim_{\varepsilon\to 0^+} \{\tau(\varepsilon) - \tau(s_2)\} \log\left|\frac{\sin\frac{1}{2}(t+\varepsilon)}{\sin\frac{1}{2}(t-\varepsilon)}\right| = 0,$$

which is itself a consequence of

$$\lim_{\varepsilon\to 0^+} \varepsilon\tau(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \log\left|\frac{\sin\frac{1}{2}(t+\varepsilon)}{\sin\frac{1}{2}(t-\varepsilon)}\right| = \cot\frac{t}{2}.$$

One also has to take into account that the integral in (3.29) is one in the sense of a Cauchy Principal Value at t, but this fact does not create difficulties.

We have thus proved that, for all $t \in (0, t_1)$,

$$\theta(t) \ge -\int_0^{s_2} K(t,s)\tau'(s)\, ds,$$

and therefore, by (3.28),

$$\theta(t) \ge \frac{\mu}{2} \int_0^{s_2} K(t,s) \frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\, dv}\, ds, \quad \text{for all } t \in (0, t_1). \quad (3.30)$$

Since s_2 was assumed to satisfy $0 < s_2 \le \pi/4$, it follows from (3.10) and (3.30) that

$$\theta(t) \ge \frac{\mu\pi}{8} \int_0^{s_2} k(t,s) \frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\, dv}\, ds, \quad \text{for all } t \in (0, t_1).$$

Let $t_2 := t_1/2$. Lemma 3.6 applied with $I := (0, t_2]$ and $\nu := 0$ yields the existence of $m_2 > 0$ such that $\theta(t) \geq m_2$ for all $t \in (0, t_2)$. This completes the proof of Proposition 3.10.

3.2.2 Limiting Equation - Proof of Theorem 3.11

Let $\varepsilon_n \to 0^+$ be such that $\theta(\varepsilon_n) \to a$ as $n \to \infty$. Proposition 3.10 shows that $a > 0$. Let $m_2 > 0$ and $t_2 > 0$ be as in the proof of Proposition 3.10, so that $\theta(t) \geq m_2$ for all $t \in (0, t_2)$. Since (3.24) holds, there exists s_3 with $0 < s_3 < t_2$ such that

$$E\Big(F^{-1}\Big(\int_0^s \sin\theta(v)\, dv\Big)\Big) \leq 2\mu, \qquad \text{for all } s \in (0, s_3). \tag{3.31}$$

As in the proof of Proposition 3.10, one can write, for all $t \in (0, s_3)$,

$$\begin{aligned} \theta(t) &= \int_0^{s_3} K(t,s)\, E\Big(F^{-1}\Big(\int_0^s \sin\theta(v)\, dv\Big)\Big) \frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\, dv}\, ds \\ &\quad + \frac{1}{2\pi} \int_{s_3}^{\pi} \Big\{ \cot\Big(\frac{1}{2}(s-t)\Big) - \cot\Big(\frac{1}{2}(s+t)\Big) \Big\} \{\tau(s_3) - \tau(s)\}\, ds \\ &=: P(t) + Q(t). \end{aligned}$$

Let $y_0 := \tan(s_3/2)$, and ϕ, R, $S : (0, y_0) \to \mathbb{R}$ be given by $\phi(x) := \theta(2 \arctan x)$, $R(x) := P(2 \arctan x)$ and $S(x) := Q(2 \arctan x)$ for all $x \in (0, y_0)$, so that

$$\phi(x) = R(x) + S(x) \qquad \text{for all } x \in (0, y_0). \tag{3.32}$$

It is clear that $\lim_{t \to 0^+} Q(t) = 0$, hence $\lim_{x \to 0^+} S(x) = 0$.

Using the identity (3.3), the change of variables $s = 2 \arctan y$ in the integral defining R leads to

$$R(x) = \int_0^{\infty} k(x,y) \frac{1}{y} \rho(y)\, dy \qquad \text{for all } x \in (0, y_0),$$

with

$$\rho(y) := \iota(2 \arctan y) \frac{\sin\theta(2 \arctan y)}{\int_0^{2 \arctan y} \sin\theta(v)\, dv} \frac{2y}{1+y^2}\, \chi_{(0,y_0)}(y),$$

where $\iota(z) := E(F^{-1}(\int_0^z \sin\theta(v)\, dv))$ for $z \in (0, s_3)$, and $\chi_{(c,d)}$ denotes the characteristic function of the interval (c, d), for $0 \leq c < d < \infty$. Since $\theta \geq m_2$ on

$(0, s_3)$ and (3.31) holds, it follows that, for all $y \in (0, \infty)$

$$|\rho(y)| \leq 2\mu \frac{1}{2 \sin m_2 \arctan y} \frac{2y}{1+y^2} \leq M,$$

where M does not depend on $y \in (0, \infty)$. Hence $\rho \in L^\infty(0, \infty)$.

Note also that

$$\rho(y) = \iota(2 \arctan y) \frac{\sin \phi(y)}{\displaystyle\int_0^y \sin \phi(u) \frac{1}{1+u^2}\, du} \frac{y}{1+y^2}\, \chi_{(0,y_0)}(y).$$

Let $\sigma : (0, \infty) \to \mathbb{R}$ be defined by

$$\sigma(x) = \int_0^\infty k(x, y) \frac{1}{y} \rho(y)\, dy \qquad \text{for all } x \in (0, \infty), \tag{3.33}$$

so that σ is an extension of R, and let $\epsilon_n := \tan(\varepsilon_n/2)$, so that $\phi(\epsilon_n) \to a$. Lemma 3.2 and Corollary 3.4 show that $\sigma \in C_b(0, \infty)$ and the sequence of functions $\{\sigma_n\}_{n \geq 1}$ given by $\sigma_n(x) := \sigma(\epsilon_n x)$ for all $x \in (0, \infty)$ is uniformly bounded and equicontinuous on $(0, \infty)$. By considering an expanding sequence of compact intervals whose union is $(0, \infty)$, the Ascoli-Arzela Theorem and a diagonalization argument yield a subsequence of $\{\sigma_n\}_{n \geq 1}$ which converges uniformly on any compact subset of $(0, \infty)$ to a function $\phi^* \in C_b(0, \infty)$. For convenience, we use for this convergent subsequence the same notation as for the original sequence. Hence, for all $x \in (0, \infty)$,

$$\phi^*(x) = \lim_{n \to \infty} \sigma_n(x) = \lim_{n \to \infty} \sigma(\epsilon_n x). \tag{3.34}$$

Let $\phi_n : (0, y_0/\epsilon_n) \to \mathbb{R}$ be defined by $\phi_n(x) := \phi(\epsilon_n x)$ for all $x \in (0, y_0/\epsilon_n)$, and $\rho_n : (0, \infty) \to \mathbb{R}$, $\rho_n(x) := \rho(\epsilon_n x)$ for all $x \in (0, \infty)$. Since $R(z) = \sigma(z)$ for all $z \in (0, y_0)$ and $\lim_{z \to 0^+} S(z) = 0$, it follows from (3.32) that, for all $x \in (0, \infty)$,

$$\lim_{n \to \infty} \phi_n(x) = \lim_{n \to \infty} (R(\epsilon_n x) + S(\epsilon_n x)) = \lim_{n \to \infty} \sigma(\epsilon_n x) = \phi^*(x). \tag{3.35}$$

Fix $x \in (0, \infty)$. A change of variables in (3.33) shows that, for all $n \geq 1$,

$$\sigma_n(x) = \int_0^\infty k(x, y) \frac{1}{y} \rho_n(y)\, dy. \tag{3.36}$$

We now want to pass to the limit in (3.36) using the Dominated Convergence Theorem. It is clear that

$$\left|k(x,y)\frac{1}{y}\rho_n(y)\right| \leq k(x,y)\frac{1}{y}||\rho||_{L^\infty(0,\infty)}, \qquad \forall n \geq 1.$$

Since, by (3.6),

$$\int_0^\infty k(x,y)\frac{1}{y}\,dy \;<\; \infty,$$

it remains to examine the pointwise convergence of the integrands. For each $y \in (0,\infty)$ and each $n \geq 1$, we have

$$\rho_n(y) := \iota(2\arctan(\epsilon_n y))\frac{\sin\phi_n(y)}{\displaystyle\int_0^y \sin\phi_n(u)\frac{1}{1+\epsilon_n^2u^2}\,du}\frac{y}{1+\epsilon_n^2y^2}\,\chi_{(0,y_0/\epsilon_n)}(y).$$

It follows from (3.24) and (3.35) that

$$\lim_{n\to\infty}\rho_n(y) = \mu\,\frac{y\sin\phi^*(y)}{\int_0^y \sin\phi^*(u)\,du} \qquad \text{for all } y \in (0,\infty).$$

Hence, passing to the limit in (3.36) taking also into account (3.34) yields that ϕ^* satisfies (3.27a). It is obvious that (3.27b) also holds, and (3.35) shows that $\phi^*(1) = \lim_{n\to\infty}\phi(\epsilon_n) = a$. This completes the proof of Theorem 3.11.

3.2.3 Uniqueness Result - Proof of Theorem 3.12

Note first that, by (3.6), the function $\phi^* \equiv \mu\pi/2$ is indeed a solution of (3.27). To show that it is the only solution, let ϕ^* satisfy (3.27), and consider the following quantities

$$l := \inf_{x\in(0,\infty)}\phi^*(x), \;\; m := \sup_{x\in(0,\infty)}\frac{1}{x}\int_0^x \phi^*(u)\,du, \;\text{ and } p := \frac{l}{m}. \tag{3.37}$$

Clearly $0 < p \leq 1$, and ϕ^* is a constant function if and only if $p = 1$. If this is the case, then it follows from the equation that $\phi^*(x) = \mu\pi/2$, for all $x \in (0,\infty)$. To prove that $p = 1$, we use the same strategy as in [1]. We derive from the integral equation (3.27) a certain inequality, see (3.45), which must be satisfied by p, and then we prove directly, in Lemma 3.13, that the only possible value of p in the interval $(0,1]$ for which this inequality holds is $p=1$.

In addition to l, m and p defined in (3.37), let

$$J := \inf_{x\in(0,\infty)} \frac{1}{x}\int_0^x \sin\phi^*(u)\,du, \text{ and } K := \sup_{x\in(0,\infty)} \frac{1}{x}\int_0^x \sin\phi^*(u)\,du. \tag{3.38}$$

Clearly

$$J \geq \sin l, \tag{3.39}$$

and it follows from Jensen's inequality for the sine function on $(0,\pi/2]$, combined with the monotonicity of the mapping $x \mapsto (\sin x)/x$ on the same interval, that

$$\frac{K}{\sin l} \leq \frac{\sin m}{\sin l} \leq \frac{m}{l} = \frac{1}{p}. \tag{3.40}$$

As in [1], an integration in (3.27) leads to

$$\int_0^x \phi^*(z)\,dz = \mu\int_0^\infty q(x,y)\frac{\sin\phi^*(y)}{\int_0^y \sin\phi^*(u)\,du}\,dy,\ x\in(0,\infty),$$

where

$$q(x,y) = \int_0^x k(z,y)\,dz = \frac{1}{\pi}\left\{x\log\frac{x+y}{|x-y|} + y\log\frac{|x^2-y^2|}{y^2}\right\},\ x\neq y.$$

Using (3.6), it follows that

$$\int_0^x \phi^*(z)\,dz = \frac{\mu\pi}{2}x + \mu\int_0^\infty q(x,y)\frac{d}{dy}\log\left\{\frac{1}{y}\int_0^y \sin\phi^*(u)\,du\right\}dy,$$

and, upon integrating by parts, we find

$$\int_0^x \phi^*(u)\,du = \frac{\mu\pi}{2}x - \mu\int_0^\infty q_y(x,y)\log\left\{\frac{1}{y}\int_0^y \sin\phi^*(u)\,du\right\}dy,$$

where

$$q_y(x,y) = \frac{1}{\pi}\log\frac{|x^2-y^2|}{y^2},\ x\neq y.$$

Since $(x-\sqrt{2}y)q_y(x,y) > 0$, for $y \neq x$, we get the estimate

$$\begin{aligned}\frac{1}{x}\int_0^x \phi^*(u)\,du &\le \frac{\mu\pi}{2} + \mu\frac{q(x,x/\sqrt{2})}{x}\log\frac{K}{J} \\ &= \frac{\mu\pi}{2} + \frac{2\mu}{\pi}\log(1+\sqrt{2})\log\frac{K}{J} \qquad \text{for all } x \in (0,\infty).\end{aligned}$$

It follows from (3.39) and (3.40) that

$$m \le \frac{2\mu}{\pi}\left\{\frac{\pi^2}{4} - \log p\,\log(1+\sqrt{2})\right\}. \tag{3.41}$$

Observe now that (3.27) can also be written in the form

$$\phi^*(x) = \mu\int_0^\infty k(x,y)\frac{d}{dy}\log\left\{\int_0^y \sin\phi^*(u)\,du\right\}dy, \;\; x \in (0,\infty),$$

and an integration by parts, as in [1], leads to

$$\phi^*(x) = \frac{\mu}{\pi}\int_0^\infty \frac{1}{u}\log\left\{1 + \frac{\int_{x|1-u|}^{x(1+u)}\sin\phi^*(v)\,dv}{\int_0^{x|1-u|}\sin\phi^*(v)\,dv}\right\}du, \;\; x \in (0,\infty). \tag{3.42}$$

It is clear from (3.37), (3.38) and (3.40) that

$$\begin{aligned}&\text{for } u \le 1: \frac{\int_{x(1-u)}^{x(1+u)}\sin\phi^*(v)\,dv}{\int_0^{x(1-u)}\sin\phi^*(v)\,dv} \ge \frac{2u}{1-u}\frac{\sin l}{K} \ge \frac{2pu}{1-u}, \\ &\text{for } u \ge 1: \frac{\int_{x(u-1)}^{x(u+1)}\sin\phi^*(v)\,dv}{\int_0^{x(u-1)}\sin\phi^*(v)\,dv} \ge \frac{2}{u-1}\frac{\sin l}{K} \ge \frac{2p}{u-1}.\end{aligned}$$

A substitution of these estimates into (3.42) gives, for all $x \in (0,\infty)$,

$$\phi^*(x) \ge \frac{\mu}{\pi}\left(\int_0^1 \frac{1}{u}\log\left\{1+\frac{2pu}{1-u}\right\}du + \int_1^\infty \frac{1}{u}\log\left\{1+\frac{2p}{u-1}\right\}du\right).$$

Since the two integrals above are equal, it follows that

$$l \ge \frac{2\mu}{\pi}\int_0^1 \frac{1}{u}\log\left\{1+\frac{2pu}{1-u}\right\}du. \tag{3.43}$$

Combining (3.41) and (3.43) gives

$$p\left\{\frac{\pi^2}{4}-\log p\,\log(1+\sqrt{2})\right\}\geq\int_0^1\frac{1}{u}\log\left\{1+\frac{2pu}{1-u}\right\}du. \tag{3.44}$$

But for every $a>0$, the change of variables formula shows that

$$\begin{aligned}\int_0^1\frac{1}{u}\log\left\{1+\frac{au}{1-u}\right\}du&=\int_0^1\frac{\log\{1+(a-1)u\}}{u}du-\int_0^1\frac{\log(1-u)}{u}du\\&=\int_0^{a-1}\frac{\log(1+v)}{v}dv+\int_0^1\frac{\log s}{s-1}ds=\int_0^a\frac{\log s}{s-1}ds.\end{aligned}$$

Thus, (3.44) is equivalent to

$$\int_0^{2p}\frac{\log s}{s-1}ds+p\log p\,\log(1+\sqrt{2})-\frac{\pi^2}{4}p\leq 0. \tag{3.45}$$

The following lemma shows that this is true only if $p=1$, and this completes the uniqueness proof.

Lemma 3.13. *Let $g:(0,1]\to\mathbb{R}$ be defined by*

$$g(x)=\int_0^{2x}\frac{\log s}{s-1}ds+x\log x\log(1+\sqrt{2})-\frac{\pi^2}{4}x,\quad x\in(0,1].$$

Then $g(1)=0$ and $g(x)>0$ for all $x\in(0,1)$.

Proof of Lemma 3.13. To show that $g(1)=0$, we use the change of variables formula to get

$$\begin{aligned}\int_0^2\frac{\log s}{s-1}ds&=\int_0^1\frac{\log s}{s-1}ds+\int_1^2\frac{\log s}{s-1}ds\\&=-\int_0^1\frac{\log(1-t)}{t}dt+\int_0^1\frac{\log(1+t)}{t}dt=\int_0^1\frac{1}{t}\log\frac{1+t}{1-t}dt.\end{aligned}$$

It follows from (3.4) that $g(1)=0$. Note also that $\lim_{x\to0^+}g(x)=0$.

We now prove that $g(x)>0$ for all $x\in(0,1)$. Since the singularity at $s=1$ of the function

$$s\mapsto\frac{\log(1+(s-1))}{s-1}$$

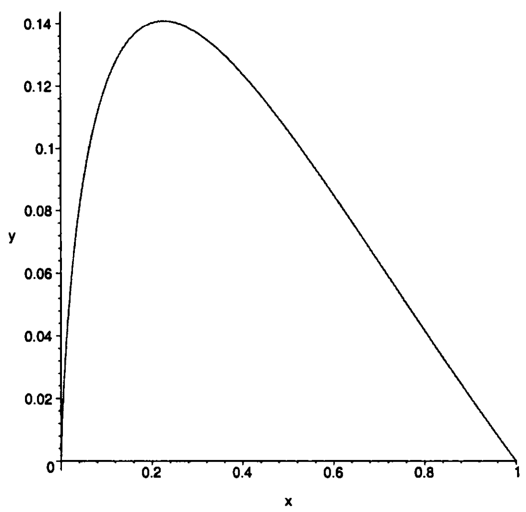

Figure 3-1: Graph of g on the interval $[0, 1]$

is removable, it follows that g is a real-analytic function on $(0, 1]$. Its first and second derivative are given, for $x \in (0, 1/2) \cup (1/2, 1]$, by

$$g'(x) = \frac{2\log(2x)}{2x-1} + \log x \log(1+\sqrt{2}) + \log(1+\sqrt{2}) - \frac{\pi^2}{4},$$

and

$$g''(x) = \frac{1}{x(2x-1)^2}\Big\{2(2x-1) - 4x\log(2x) + (2x-1)^2\log(1+\sqrt{2})\Big\}. \quad (3.46)$$

Let $j : (0, 1] \to \mathbb{R}$ be given by

$$j(x) = 2(2x-1) - 4x\log(2x) + (2x-1)^2\log(1+\sqrt{2}), \quad x \in (0, 1]. \quad (3.47)$$

Then

$$j'(x) = -4\log(2x) + 4(2x-1)\log(1+\sqrt{2}), \quad x \in (0, 1],$$
$$j''(x) = -\frac{4}{x} + 8\log(1+\sqrt{2}), \quad x \in (0, 1].$$

Let $x_0 = (2\log(1+\sqrt{2}))^{-1}$, so that $j'' < 0$ on $(0, x_0)$ and $j'' > 0$ on $(x_0, 1]$. Note that $x_0 \in (1/2, 1)$. Since j' is strictly decreasing on $(0, x_0)$, strictly increasing on

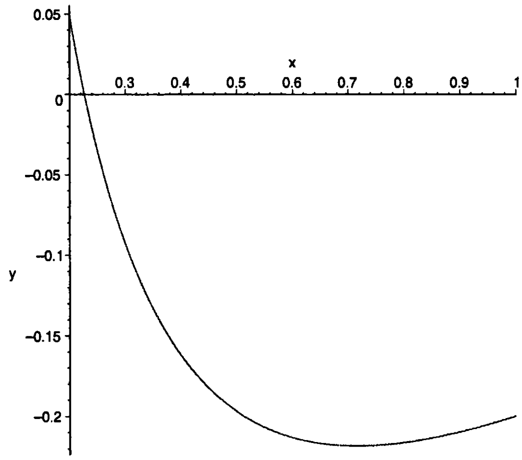

Figure 3-2: Graph of g' on the interval $[0.2, 1]$

$(x_0, 1)$, and

$$\lim_{x \to 0+} j'(x) = +\infty, \; j'(1/2) = 0, \; j'(1) = -4\log 2 + 4\log(1+\sqrt{2}) > 0,$$

it follows that there exists $x_1 \in (x_0, 1)$ such that $j' > 0$ on $(0, 1/2) \cup (x_1, 1)$ and $j' < 0$ on $(1/2, x_1)$. Hence j is strictly increasing on each of the intervals $(0, 1/2)$ and $(x_1, 1)$, and strictly decreasing on $(1/2, x_1)$. Since $j(1/2) = 0$,

$$\lim_{x \to 0+} j(x) = -2 + \log(1+\sqrt{2}) < 0, \text{ and } j(1) = 2 - 4\log 2 + \log(1+\sqrt{2}) > 0,$$

it follows that there exists $x_2 \in (x_1, 1)$ such that $j < 0$ on $(0, 1/2) \cup (1/2, x_2)$, $j(1/2) = 0$ and $j > 0$ on $(x_2, 1)$.

This shows, taking into account (3.46), (3.47) and the continuity of g'' on the interval $(0, 1)$, that g' is strictly decreasing on $(0, x_2)$ and strictly increasing on $(x_2, 1)$. Since

$$\lim_{x \to 0+} g'(x) = +\infty, \text{ and } g'(1) = 2\log 2 + \log(1+\sqrt{2}) - \frac{\pi^2}{4} < 0,$$

it follows that there exists $x_3 \in (0, x_2)$ such that $g' > 0$ on $(0, x_3)$, and $g' < 0$ on $(x_3, 1)$. Thus g is strictly increasing on $(0, x_3)$ and strictly decreasing on $(x_3, 1)$

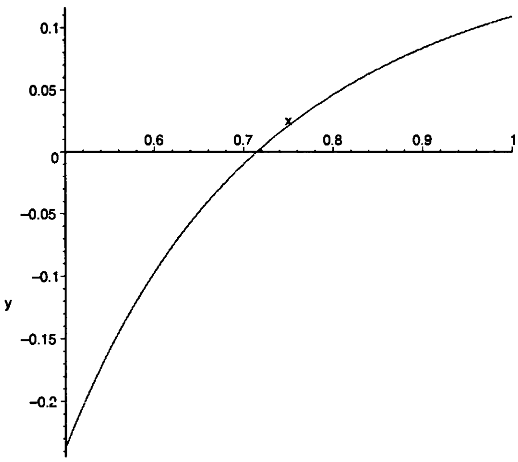

Figure 3-3: Graph of g'' on the interval $[0.5, 1]$

and, since

$$\lim_{x \to 0^+} g(x) = 0 \quad \text{and} \quad g(1) = 0,$$

it follows that $g(x) > 0$ for all $x \in (0, 1)$. This completes the proof of the lemma. $\square$

3.2.4 On Assumption (3.24)

We now explore condition (3.24), which was the key ingredient for the previous results. In this subsection $f : [0, \infty) \to [0, \infty)$ is continuous, $f(0) = 0$, $f > 0$ on $(0, \infty)$, $f \in C^1(0, \delta)$ for some $\delta > 0$, $F' = f$, $F(0) = 0$, $E := f'F/f^2$.

Proposition 3.14. *Let* $\mu \neq 0$ *be such that* $\lim_{r \to 0^+} E(r) = \mu$. *Then necessarily* $\mu \in (0, 1]$.

Proof. Let $\{x_n\}_{n \geq 1}$ be a sequence in $(0, \delta)$ with $x_n \to 0^+$ as $n \to \infty$. Since $f(0) = 0$ and $f(x_n) > 0$, the Mean Value Theorem shows that, for all $n \geq 1$, there exists $\xi_n \in (0, x_n)$ such that $f'(\xi_n) > 0$. Since $\lim_{r \to 0^+} E(r) = \mu \neq 0$, it follows that $\mu > 0$.

This shows that $f'(r) > 0$ for all r sufficiently small, and there is no loss of generality in assuming that $f' > 0$ on $(0, \delta)$. Hence the function f is strictly increasing on $(0, \delta)$, and hence $F(r) \leq rf(r)$ for all $r \in (0, \delta)$. It follows that the function $g : [0, \delta) \to \mathbb{R}$ given by $g(0) = 0$ and $g = F/f$ on $(0, \delta)$ is continuous on

$[0, \delta)$ and differentiable on $(0, \delta)$, with derivative given by

$$g'(r) = \frac{f^2(r) - f'(r)F(r)}{f^2(r)} = 1 - E(r) \quad \text{for all } r \in (0, \delta).$$

Let $\{x_n\}_{n \geq 1}$ be a sequence in $(0, \delta)$ with $x_n \to 0^+$ as $n \to \infty$. Since $g(0) = 0$ and $g(x_n) > 0$, the Mean Value Theorem shows that, for all $n \geq 1$, there exists $\xi_n \in (0, x_n)$ such that $g'(\xi_n) > 0$. Since $\lim_{r \to 0^+} g'(r) = 1 - \mu$, it follows that $\mu \leq 1$. This completes the proof. □

We now examine conditions which are either necessary or sufficient for (3.24).

Suppose first that (3.24) holds. In the notation used in the proof of the previous Proposition one has, since $\lim_{r \to 0^+} g'(r) = 1 - \mu$, that $\lim_{r \to 0^+} g(r)/r = 1 - \mu$. This means that

$$\lim_{r \to 0^+} \frac{F(r)}{rf(r)} = 1 - \mu.$$

A sufficient condition for (3.24) to hold is that there exists $c > 0$ and $\alpha > 0$ such that

$$\lim_{r \to 0^+} \frac{f'(r)}{f^{1-\frac{1}{\alpha}}(r)} = c. \tag{3.48}$$

Indeed, if (3.48) holds, then it follows from L'Hospital's Theorem that

$$\lim_{r \to 0^+} \frac{f^{1+\frac{1}{\alpha}}(r)}{F(r)} = \frac{\alpha + 1}{\alpha} \lim_{r \to 0^+} \frac{f^{1/\alpha}(r) f'(r)}{f(r)} = \frac{\alpha + 1}{\alpha} c. \tag{3.49}$$

Combining (3.48) and (3.49) yields (3.24) with $\mu := \alpha/(\alpha + 1)$.

In turn, a sufficient condition for (3.48) is that there exists $C > 0$ and $\alpha > 0$ such that $\lim_{r \to 0^+} f'(r)/r^{\alpha - 1} = C$. Indeed, if this holds, then L'Hospital Theorem shows that $\lim_{r \to 0^+} \frac{f(r)}{r^\alpha} = \frac{C}{\alpha}$, and therefore (3.48) holds with $c := C^{\frac{1}{\alpha}} \alpha^{1 - \frac{1}{\alpha}}$.

3.2.5 An Example

We now construct an example of a function $\lambda : [a, 0] \to [0, \infty)$, for some $a < 0$, such that λ is continuous on $[a, 0]$ and real-analytic on $(a, 0)$, and a solution $(\mathcal{S}, \psi)$ of (1.1) such that $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic and even, with $\eta(0) = 0$, $\eta(\pi) = a$ and η strictly decreasing on $[0, \pi]$, but such that $\mathcal{S}$ does not have a corner at the origin, i.e. $\lim_{X \to 0^+} \eta'(X)$ does not exist.

Let $\alpha \in (0,1)$, $\mathcal{A} := \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta < 1\}$, and let the holomorphic function $\widetilde{V} : \mathcal{A} \to \mathbb{C}$ be given by

$$\widetilde{V}(\zeta) := -\frac{\kappa}{\alpha}(1-\zeta)^{\alpha}\Big[1 + \varepsilon\cos\big(\alpha\log(1-\zeta)\big)\Big] \quad \text{for all } \zeta \in \mathcal{A},$$

where $\kappa > 0$, $\varepsilon > 0$, and we choose $\arg(1-\zeta) \in (-\pi/2, \pi/2)$ for $\zeta \in \mathcal{A}$. If we put $\widetilde{V}(1) = 0$, then $\widetilde{V}$ becomes a continuous function in the closed unit disc $\overline{D}$. Let $w : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that

$$\widetilde{V}(e^{it}) = w(t) + i\mathcal{C}w(t) \quad \text{for all } t \in \mathbb{R}.$$

Then w is continuous and even, $\mathcal{C}w$ is continuous and odd, and w, $\mathcal{C}w$ are real-analytic on $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$, with classical derivatives given by

$$w'(t) + i(\mathcal{C}w)'(t) = \frac{d}{dt}\{\widetilde{V}(e^{it})\} = ie^{it}\widetilde{V}'(e^{it}) \quad \text{for all } t \in (0,\pi],$$

where $\widetilde{V}'$ denotes the complex derivative of $\widetilde{V}$. Let $\widetilde{W} : \mathcal{A} \to \mathbb{C}$ be given by $\widetilde{W}(\zeta) = 1 + \zeta\widetilde{V}'(\zeta)$ for all $\zeta \in \mathcal{A}$, so that

$$\widetilde{W}(\zeta) = 1 + \kappa\frac{\zeta}{(1-\zeta)^{\beta}}\Big[1 + \varepsilon\sqrt{2}\cos\Big(\frac{\pi}{4} + \alpha\log(1-\zeta)\Big)\Big] \quad \text{for all } \zeta \in \mathcal{A},$$

where $\beta := 1 - \alpha$. Let $\Gamma : (0,\pi] \to \mathbb{C}$ be given by

$$\Gamma(t) := 1 + (\mathcal{C}w)'(t) - iw'(t) = \widetilde{W}(e^{it}), \text{ for all } t \in (0,\pi].$$

Then, for all $t \in (0,\pi]$,

$$\begin{aligned}\Gamma(t) = 1 + \kappa&\frac{\cos\big(t + \beta\frac{\pi-t}{2}\big) + i\sin\big(t + \beta\frac{\pi-t}{2}\big)}{(2\sin\frac{t}{2})^{\beta}}\times \\ &\times\Bigg\{1 + \varepsilon\sqrt{2}\Bigg[\cos\Big(\frac{\pi}{4} + \alpha\log\Big(2\sin\frac{t}{2}\Big)\Big)\cosh\Big(\alpha\frac{\pi-t}{2}\Big) + \\ &\qquad + i\sin\Big(\frac{\pi}{4} + \alpha\log\Big(2\sin\frac{t}{2}\Big)\Big)\sinh\Big(\alpha\frac{\pi-t}{2}\Big)\Bigg]\Bigg\}.\end{aligned} \tag{3.50}$$

It is clear from (3.50) that, if ε and κ are chosen sufficiently small, then

$$1+(\mathcal{C}w)'(t) \geq c > 0 \quad \text{and} \quad w'(t) < 0 \quad \text{for all } t \in (0,\pi), \tag{3.51}$$

$$|\Gamma(t)| \to +\infty \quad \text{as} \quad t \to 0^+, \tag{3.52}$$

It follows from (3.51) that the continuous periodic functions w and $\mathcal{C}w$ have bounded variation on $[-\pi,\pi]$. Hence, by [5, Section 3.4, p. 42] and [35, Lemma 2.2], w and $\mathcal{C}w$ are absolutely continuous, $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ and $(\mathcal{C}w)' = \mathcal{C}w'$, $\widetilde{W} \in \mathcal{H}^1_{\mathbb{C}}$ and $\widetilde{W}^* = 1+\mathcal{C}w' - iw'$ almost everywhere. Let $W := i\widetilde{W}$, so that (1.25a) holds.

Let $a := w(\pi)$. Since (3.51) holds, a function $\lambda : [a,0) \to (0,\infty)$ can be unambiguously defined by

$$\lambda(w(t)) := \frac{1}{w'(t)^2 + (1+\mathcal{C}w'(t))^2} \quad \text{for all } t \in (0,\pi]. \tag{3.53}$$

Clearly λ is real-analytic on $(a,0)$ and continuous on $[a,0)$. Moreover, if we put $\lambda(0) := 0$, then (3.52) shows that λ is continuous on $[a,0]$.

Since $\widetilde{W} \in \mathcal{H}^1_{\mathbb{C}}$ and $\operatorname{Re}\widetilde{W}^* \geq c > 0$ almost everywhere, it follows that $\operatorname{Re}\widetilde{W}(\zeta) \geq c$ for all $\zeta \in D$, and hence $1/\widetilde{W} \in \mathcal{H}^\infty_{\mathbb{C}}$. Therefore W satisfies (1.25b), while the definition of λ in (3.53) and the evenness and periodicity of w ensure that (1.25c) holds. As shown in Chapter 1, the functions τ, ϑ defined by (1.29) satisfy (1.30), and hence ϑ satisfies the generalized Nekrasov's equation (1.33). It is clear from (3.51) that (1.24) holds, and hence the curve

$$\mathcal{S} := \{(-(t+\mathcal{C}w(t)), w(t)) : t \in \mathbb{R}\}$$

describes a free boundary. If $\theta := -\vartheta$, (1.30) shows that θ gives the angle between the tangent to $\mathcal{S}$ and the horizontal, and $\theta(t) = \arg\Gamma(t)$ for all $t \in (0,\pi]$, where Γ is given by (3.50).

Consider now sequences $\{t_n^\pm\}_{n\geq 1}$ with $t_n^\pm \searrow 0$ as $n \to \infty$, such that

$$\cos\Big(\frac{\pi}{4} + \alpha\log\Big(2\sin\frac{t_n^\pm}{2}\Big)\Big) + i\sin\Big(\frac{\pi}{4} + \alpha\log\Big(2\sin\frac{t_n^\pm}{2}\Big)\Big) = \pm i, \text{ for all } n \geq 1.$$

It is then clear from (3.50) that, as $n \to \infty$,

$$\theta(t_n^{\pm}) = \arg\Gamma(t_n^{\pm}) \longrightarrow \frac{\beta\pi}{2} \pm \arctan\left(\varepsilon\sqrt{2}\sinh\left(\frac{\alpha\pi}{2}\right)\right),$$

hence $\theta(t)$ does not have a limit as $t \searrow 0$.

3.3 Asymmetric Singular Free Boundaries

We now consider a solution $(\mathcal{S}, \psi)$ of (1.1) for which $(0,0)$ is an isolated stagnation point which is not necessarily symmetric. We show in Theorem 3.15 below that, if there are lateral tangents to the free boundary at the stagnation point then, under natural assumptions on h, they must be symmetric with respect to the vertical line passing through that point, and enclose an angle of the expected size. This result is not as strong as its counterpart, Theorem 3.7, for symmetric free boundaries, where the existence of lateral tangents was not assumed, but deduced. It remains an open problem whether, under suitable assumptions on h, free boundaries must necessarily have corners at isolated asymmetric stagnation points.

Theorem 3.15. *Let h be as in* (1.4) *and such that, for some $\delta > 0$, the restriction of h to $(-\delta, 0)$ is of class C^1. With E as in* (3.23)*, suppose that* (3.24) *holds and, is addition, suppose that*

$$\lim_{r \to 0^+} \frac{f(r)}{F^{\mu}(r)} = c \quad \textit{for some } c \in (0, \infty). \tag{3.54}$$

Let $(\mathcal{S}, \psi)$ be a solution of (1.1)*, where $\mathcal{S}$ contains the point $(0,0)$. Suppose that there exists $X_0 \in (0, \pi]$, $Y_0 < 0$ and $\eta : [-X_0, X_0] \to \mathbb{R}$ a continuous function, strictly increasing on $[-X_0, 0]$, strictly decreasing on $[0, X_0]$, with $\eta(0) = 0$, and such that*

$$\mathcal{S} \cap \mathcal{R} = \{(X, \eta(X)) : -X_0 \le X \le X_0\},$$

where $\mathcal{R}$ denotes the rectangle $\{(X,Y) : -X_0 \le X \le X_0, Y_0 \le Y \le 0\}$. In addition, suppose that η is continuously differentiable on $[-X_0, 0) \cup (0, X_0]$, and *there exist*

$$\lim_{X\nearrow 0}\eta'(X)\in[0,+\infty] \quad \textit{and} \quad \lim_{X\searrow 0}\eta'(X)\in[-\infty,0].$$

Then

$$\lim_{X\to\pm 0}\eta'(X)=\mp\tan(\mu\pi/2),$$

where μ *is given by* (3.24).

As seen at the beginning of Section 3.2, free boundaries $\mathcal{S}$ as in Theorem 3.15 can be described by solutions θ of (3.26). The assumption that $\mathcal{S}$ has lateral tangents at the origin means that

$$\lim_{t\searrow 0}\theta(t)=2n\pi+a \text{ and } \lim_{t\nearrow 0}\theta(t)=2n\pi-b \quad \text{for some } a,b\in[0,\pi/2]. \tag{3.55}$$

Theorem 3.15 will be proved once we show that Theorem 3.16 below holds.

Theorem 3.16. *Suppose that* f *satisfies* (3.24) *and* (3.54). *Let* θ *be a solution of* (3.26) *satisfying* (3.55). *Then* $a=b=\mu\pi/2$.

We observe first how, from a solution θ of (3.26), one can construct another solution $\tilde{\theta}$ of (3.26) by reflection.

For any 2π-periodic function u, let Ru be the 2π-periodic function given by $Ru(t)=u(-t)$, for all $t\in\mathbb{R}$. It is easy to see from the definition of a conjugate function (1.14) that, if $u\in L^1_{2\pi}$, then $\mathcal{C}(Ru)=-R(\mathcal{C}u)$. This observation can be used to show that, if (θ,τ) satisfy (3.26) then $(\tilde{\theta},\tilde{\tau})$ is another solution of (3.26), where

$$\tilde{\theta}:=-R\theta, \qquad \tilde{\tau}:=R\tau \quad \text{and} \quad \tilde{n}:=-n. \tag{3.56}$$

Define

$$\hat{\theta}:=(\theta+\tilde{\theta})/2, \qquad \hat{\tau}:=(\tau+\tilde{\tau})/2. \tag{3.57}$$

Then $\hat{\theta}$ is an odd function, $\hat{\tau}$ is an even function, and

$$\hat{\theta}=\mathcal{C}\hat{\tau}. \tag{3.58}$$

The first step in the proof of Theorem 3.16 is the following result.

Proposition 3.17. *Let θ satisfy* (3.26), *where* (3.24) *holds. Then*

$$\liminf_{t\searrow 0}\,(\theta(t)-\theta(-t))>0. \tag{3.59}$$

Note that in Proposition 3.17 we assume neither that f satisfies (3.54), nor that θ satisfies (3.55). Proposition 3.17 is a full extension to the nonsymmetric case of Proposition 3.10.

Proof of Proposition 3.17. Let $\tilde\theta$, $\tilde\tau$, $\hat\theta$, $\hat\tau$ be defined as in (3.56) and (3.57). We aim to prove that $\liminf_{t\searrow 0}\hat\theta(t)>0$.

Let $s_0\in(0,t_0)$ be sufficiently small so that $F^{-1}(\int_0^s\sin\theta(v)\,dv)<\delta$ and $F^{-1}(\int_0^s\sin\tilde\theta(v)\,dv)<\delta$ for all $s\in(0,s_0)$. Then $\hat\tau$ is a continuously differentiable function on $(0,s_0)$ and, for all $s\in(0,s_0)$,

$$\begin{aligned}-2\hat\tau'(s)&=E\Big(F^{-1}\Big(\int_0^s\sin\theta(v)\,dv\Big)\Big)\frac{\sin\theta(s)}{\int_0^s\sin\theta(v)\,dv}\\&\quad+E\Big(F^{-1}\Big(\int_0^s\sin\tilde\theta(v)\,dv\Big)\Big)\frac{\sin\tilde\theta(s)}{\int_0^s\sin\tilde\theta(v)\,dv}.\end{aligned}$$

It follows from (3.24) that there exists s_1 with $0<s_1<s_0$ such that $\hat\tau$ is a decreasing function on $(0,s_1)$ and, moreover,

$$-\hat\tau'(s)\geq\frac{\mu}{4}\left(\frac{\sin\theta(s)}{\int_0^s\sin\theta(v)\,dv}+\frac{\sin\tilde\theta(s)}{\int_0^s\sin\tilde\theta(v)\,dv}\right)\quad\text{for all } s\in(0,s_1). \tag{3.60}$$

Since $\hat\tau\in L^1_{2\pi}$, the monotonicity of $\hat\tau$ shows that $\int_0^s\hat\tau(v)\,dv\geq s\hat\tau(s)\geq 0$ for all $s\in(0,s_1)$, which implies that $\lim_{s\to 0^+}s\hat\tau(s)=0$.

Since $\hat\tau$ is an even function, the definition of a conjugate function shows that, for any s_2 with $0<s_2<s_1$, and for all $t\in(0,s_2)$,

$$\begin{aligned}\hat\theta(t)&=\frac{1}{2\pi}\int_0^{s_2}\Big\{\cot\Big(\frac{1}{2}(s+t)\Big)-\cot\Big(\frac{1}{2}(s-t)\Big)\Big\}\{\hat\tau(s)-\hat\tau(s_2)\}\,ds\\&\quad+\frac{1}{2\pi}\int_{s_2}^{\pi}\Big\{\cot\Big(\frac{1}{2}(s-t)\Big)-\cot\Big(\frac{1}{2}(s+t)\Big)\Big\}\{\hat\tau(s_2)-\hat\tau(s)\}\,ds\\&=:\widehat P_{s_2}(t)+\widehat Q_{s_2}(t).\end{aligned}$$

Since $\hat\tau$ is decreasing on $(0,s_1)$, $\hat\tau\in L^1_{2\pi}$ and $\hat\tau(s)\to\infty$ as $s\to 0^+$, we get, arguing as in the proof of Proposition 3.10, that there exist s_2 sufficiently small, with $s_2 \in (0, \pi/4]$, and t_1 with $0 < t_1 < s_2$ such that $\widehat{Q}_{s_2}(t) \geq 0$ for all $t \in (0, t_1)$.

For any $t \in (0, s_2)$, one can integrate by parts in the formula giving $\widehat{P}_{s_2}(t)$, as in the proof of Proposition 3.10, using now the fact that $\lim_{s \to 0^+} s\hat{r}(s) = 0$, to conclude that

$$\widehat{P}_{s_2}(t) = -\int_0^{s_2} K(t,s)\hat{r}'(s)\, ds,$$

where K is given by (3.1).

We have thus proved that, for all $t \in (0, t_1)$,

$$\hat{\theta}(t) \geq -\int_0^{s_2} K(t,s)\hat{r}'(s)\, ds,$$

and therefore, by (3.60), for all $t \in (0, t_1)$,

$$\hat{\theta}(t) \geq \frac{\mu}{4}\int_0^{s_2} K(t,s)\left(\frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\, dv} + \frac{\sin\tilde{\theta}(s)}{\int_0^s \sin\tilde{\theta}(v)\, dv}\right) ds. \tag{3.61}$$

Since s_2 was assumed to satisfy $0 < s_2 \leq \pi/4$, it follows from (3.10) and (3.61) that, for all $t \in (0, t_1)$,

$$\hat{\theta}(t) \geq \frac{\mu\pi}{16}\int_0^{s_2} k(t,s)\left(\frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\, dv} + \frac{\sin\tilde{\theta}(s)}{\int_0^s \sin\tilde{\theta}(v)\, dv}\right) ds. \tag{3.62}$$

Recall now from (3.26g) and (3.56) that

$$\theta(t) \in [2n\pi, 2n\pi + \pi/2], \qquad \tilde{\theta}(t) \in [-2n\pi, -2n\pi + \pi/2] \qquad \text{for all } t \in (0, t_0). \tag{3.63}$$

Since, for $0 \leq \phi \leq \pi/2$,

$$\frac{2}{\pi}\phi \leq \sin\phi \leq \phi, \tag{3.64}$$

we deduce from (3.62) that

$$\hat{\theta}(t) \geq \frac{\mu}{8}\int_0^{s_2} k(t,s)\left(\frac{\theta(s) - 2n\pi}{\int_0^s (\theta(v) - 2n\pi)\, dv} + \frac{\tilde{\theta}(s) + 2n\pi}{\int_0^s (\tilde{\theta}(v) + 2n\pi)\, dv}\right) ds. \tag{3.65}$$

It follows from (3.65), using the trivial inequality

$$\frac{a}{b}+\frac{c}{d}\geq\frac{a+c}{b+d}\qquad b,d>0,\ a,c\geq 0,$$

that

$$\hat{\theta}(t)\geq\frac{\mu}{8}\int_0^{s_2}k(t,s)\frac{\hat{\theta}(s)}{\int_0^s\hat{\theta}(v)\,dv}\,ds. \tag{3.66}$$

Since (3.63) implies that $\hat{\theta}(t)\in[0,\pi/2]$ for all $t\in(0,t_0)$, it follows from (3.66), using again (3.64), that

$$\hat{\theta}(t)\geq\frac{\mu}{4\pi}\int_0^{s_2}k(t,s)\frac{\sin\hat{\theta}(s)}{\int_0^s\sin\hat{\theta}(v)\,dv}\,ds. \tag{3.67}$$

Let $t_2:=t_1/2$. Lemma 3.6 applied with $I:=(0,t_2]$ and $\nu:=0$ yields the existence of $m_2>0$ such that $\hat{\theta}(t)\geq m_2$ for all $t\in(0,t_2)$. This completes the proof of the Proposition. □

Proof of Theorem 3.15. It follows from Proposition 3.17 that in (3.55) at least one of $a\neq 0$ and $b\neq 0$ holds. Suppose for a contradiction that $a\neq b$. We apply Lemma 1.5 with $u:=\tau$ and $Cu:=\theta$. Note that

$$\tau(s)-\tau(-s)=\log\left\{\frac{f\Big(F^{-1}\Big(\int_0^s\sin\theta(v)\,dv\Big)\Big)}{f\Big(F^{-1}\Big(\int_0^s\sin\tilde{\theta}(v)\,dv\Big)\Big)}\right\}. \tag{3.68}$$

It is an immediate consequence of (3.54) that, as $s\to 0^+$,

$$\frac{f\Big(F^{-1}\Big(\int_0^s\sin\theta(v)\,dv\Big)\Big)}{\Big(\int_0^s\sin\theta(v)\,dv\Big)^{\mu}}\to c,\qquad\frac{f\Big(F^{-1}\Big(\int_0^s\sin\tilde{\theta}(v)\,dv\Big)\Big)}{\Big(\int_0^s\sin\tilde{\theta}(v)\,dv\Big)^{\mu}}\to c. \tag{3.69}$$

Using (3.69) and the facts that $a\neq b$, $(a,b)\neq(0,0)$, we conclude from (3.68) that either (1.16) or (1.17) holds, with $u:=\tau$. Therefore, by Lemma 1.5, θ should be unbounded on any interval containing 0. This contradicts the fact that θ is bounded on $(-t_0,t_0)$. Therefore we must necessarily have $a=b\neq 0$. It remains to show that $a=b=\mu\pi/2$.

Let $m_2>0$ and $t_2>0$ be such that $\theta(t)\geq 2n\pi+m_2$ and $\tilde{\theta}(t)\geq -2n\pi+m_2$ for all $t\in(0,t_2)$. Since (1.8) holds, there exists s_3 with $0<s_3<t_2$ such that,

for all $s \in (0, s_3)$,

$$E\Big(F^{-1}\Big(\int_0^s \sin\tilde{\theta}(v)\,dv\Big)\Big) \leq 2\mu, \qquad E\Big(F^{-1}\Big(\int_0^s \sin\tilde{\theta}(v)\,dv\Big)\Big) \leq 2\mu. \tag{3.70}$$

As in the proof of Proposition 3.10, one can write, for all $t \in (0, s_3)$,

$$\begin{aligned}
\hat{\theta}(t) = \frac{1}{2}\bigg\{ &\int_0^{s_3} K(t,s)\, E\Big(F^{-1}\Big(\int_0^s \sin\theta(v)\,dv\Big)\Big)\frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\,dv}\,ds \\
&+ \int_0^{s_3} K(t,s)\, E\Big(F^{-1}\Big(\int_0^s \sin\tilde{\theta}(v)\,dv\Big)\Big)\frac{\sin\tilde{\theta}(s)}{\int_0^s \sin\tilde{\theta}(v)\,dv}\,ds\bigg\} \\
&+ \frac{1}{2\pi}\int_{s_3}^{\pi} \Big\{\cot\Big(\frac{1}{2}(s-t)\Big) - \cot\Big(\frac{1}{2}(s+t)\Big)\Big\}\{\hat{\tau}(s_3) - \hat{\tau}(s)\}\,ds \\
=:\ & \widehat{P}(t) + \widehat{Q}(t).
\end{aligned}$$

Let $y_0 := \tan(s_3/2)$, and $\hat{\phi}$, $\widehat{R}$, $\widehat{S} : (0, y_0) \to \mathbb{R}$ be given by $\hat{\phi}(x) := \hat{\theta}(2\arctan x)$, $\widehat{R}(x) := \widehat{P}(2\arctan x)$ and $\widehat{S}(x) := Q(2\arctan x)$ for all $x \in (0, y_0)$, so that

$$\hat{\phi}(x) = \widehat{R}(x) + \widehat{S}(x) \quad \text{for all } x \in (0, y_0). \tag{3.71}$$

It is clear that $\lim_{t\to 0^+} \widehat{Q}(t) = 0$, hence $\lim_{x\to 0^+} \widehat{S}(x) = 0$.

Using the identity (3.3), the change of variables $s = 2\arctan y$ in the integral defining R leads to

$$\widehat{R}(x) = \int_0^\infty k(x,y)\frac{1}{y}\hat{\rho}(y)\,dy \quad \text{for all } x \in (0, y_0),$$

with

$$\begin{aligned}
2\hat{\rho}(y) :&= \iota(2\arctan y)\frac{\sin\theta(2\arctan y)}{\int_0^{2\arctan y}\sin\theta(v)\,dv}\,\frac{2y}{1+y^2}\,\chi_{(0,y_0)}(y), \\
&+ \tilde{\iota}(2\arctan y)\frac{\sin\tilde{\theta}(2\arctan y)}{\int_0^{2\arctan y}\sin\tilde{\theta}(v)\,dv}\,\frac{2y}{1+y^2}\,\chi_{(0,y_0)}(y)
\end{aligned} \tag{3.72}$$

where $\iota(z) := E(F^{-1}(\int_0^z \sin\theta(v)\,dv))$ and $\tilde{\iota}(z) := E(F^{-1}(\int_0^z \sin\tilde{\theta}(v)\,dv))$ for $z \in (0, s_3)$, while $\chi_{(c,d)}$ denotes the characteristic function of the interval (c,d), for $0 \leq c < d < \infty$.

Using (3.24) and the fact that $\lim_{t\to 0^+}\theta(t) = \lim_{t\to 0^+}\tilde{\theta}(t) = a \neq 0$, it is immediate from (3.72) that $\lim_{y\to 0}\hat{\rho}(y) = \mu$. Now Lemma 3.5 shows that $\lim_{x\to 0^+}\widehat{R}(x) = \mu\pi/2$. It follows from (3.71) that $\lim_{x\to 0^+}\hat{\phi}(x) = \mu\pi/2$, and hence $\lim_{t\to 0^+}\hat{\theta}(t) = \mu\pi/2$, which shows that $a = b = \mu\pi/2$, as required. This completes the proof of Theorem 3.15. $\square$

Chapter 4

The Water-Wave Problem

In this chapter we consider the water-wave problem. We show how the existence of a singular solution of Nekrasov's equation can be deduced from the existence of a suitable sequence of smooth solutions, under more general assumptions than those used by Toland [32] and McLeod [15]. We also show that a Gibbs Phenomenon, originally exhibited in [15], which occurs as smooth solutions approach a singular solution, is still displayed in the present wider setting. We also prove a new result on the asymptotic behaviour of solutions of McLeod's 'boundary layer equation'.

4.1 Existence of Singular Solutions

Theorem 4.1. *Let $\{\nu_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\nu_n \to 0$ as $n \to \infty$, and let $\{\theta_n\}_{n\geq 1}$, $\{\tau_n\}_{n\geq 1}$ be sequences of continuous, 2π-periodic functions which satisfy Nekrasov's equation*

$$\theta_n = \mathcal{C}\tau_n, \tag{4.1a}$$

$$\tau_n(s) = -\frac{1}{3}\log\left(\nu_n + \int_0^s \sin\theta_n(v)\,dv\right) + a_n, \text{ for all } s \in \mathbb{R}, \tag{4.1b}$$

$$[\tau_n] = 0. \tag{4.1c}$$

where $\{a_n\}_{n\geq 1}$ is sequence of real numbers. Suppose that all the functions θ_n are odd, and that there exists $t_0 \in (0,\pi]$ such that, for all $n \geq 1$,

$$\sin\theta_n \geq 0 \text{ on } (0,t_0) \text{ and } \int_{t_0}^t \sin\theta_n(v)\,dv \geq 0 \text{ for all } t \in (t_0,\pi]. \tag{4.2}$$

Then there exist $a \in \mathbb{R}$ and functions θ, τ continuous on $\mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}$, with θ an odd function, satisfying

$$\theta = C\tau \tag{4.3a}$$

$$\tau(s) = -\frac{1}{3}\log\Big(\int_0^s \sin\theta(v)\,dv\Big) + a, \text{ for all } s \in \mathbb{R}, \tag{4.3b}$$

$$[\tau] = 0, \tag{4.3c}$$

such that a subsequence of $\{a_n\}_{n\geq 1}$ tends to a as $n \to \infty$, and the corresponding subsequences of $\{\theta_n\}_{n\geq 1}$, $\{\tau_n\}_{n\geq 1}$ converge uniformly on any interval $[\delta, \pi]$, for $\delta \in (0, \pi]$, to θ and τ respectively.

Since τ_n is an even function for all $n \geq 1$, an integration by parts in the definition of a conjugate functions shows that θ_n satisfies the classical form of Nekrasov's equation

$$\theta_n(t) = \frac{1}{3}\int_0^\pi K(t,s)\frac{\sin\theta_n(s)}{\nu_n + \int_0^s \sin\theta_n(v)\,dv}\,ds, \qquad t \in (0, \pi], \tag{4.4}$$

where K is given by (3.1). The function θ satisfies

$$\theta(t) = \frac{1}{3}\int_0^\pi K(t,s)\frac{\sin\theta(s)}{\int_0^s \sin\theta(v)\,dv}\,ds, \qquad t \in (0, \pi].$$

With the assumption (4.2) replaced by $0 \leq \theta_n \leq \pi/2$ on $[0, \pi]$, Theorem 4.1 was proved in [32], and also in [15]. The existence of a sequence of such solutions was known from the work of Keady and Norbury [9]. The proof in [32] makes use of both formulations (4.1) and (4.4), while that in [15] uses (4.4) only.

The present proof is based on ideas in [15], with some simplification due to the use of Privalov's Theorem.

Proof of Theorem 4.1. Since $\nu_n > 0$ for each $n \geq 1$, it follows that (τ_n, θ_n) corresponds to a solution w of (1.25) for which $\mathcal{N} = \emptyset$, with a nonlinearity h which is real-analytic on $(-\infty, 0)$. Hence τ_n, θ_n are actually real-analytic functions. Note also that, by the main result of [36],

$$-\pi/2 < \theta_n < \pi/2 \quad \text{for all } n \geq 1. \tag{4.5}$$

For all $n \geq 1$, the derivative of τ_n is given by

$$\tau_n'(s) = -\frac{1}{3}\frac{\sin\theta_n(s)}{\nu_n + \int_0^s \sin\theta_n(v)\,dv} \qquad \text{for all } s \in [-\pi, 0) \cup (0, \pi]. \tag{4.6}$$

Since $\tau_n - \tau_n(t_0)$ is an even function, and $\theta_n = \mathcal{C}(\tau_n - \tau_n(t_0))$, the definition of a conjugate function yields, upon integrating by parts on $(0, t_0)$, that

$$\begin{aligned}
\theta_n(t) &= \frac{1}{3}\int_0^{t_0} K(t,s)\frac{\sin\theta_n(s)}{\nu_n + \int_0^s \sin\theta_n(v)\,dv}\,ds \\
&\quad + \frac{1}{2\pi}\int_{t_0}^{\pi}\left\{\cot\left(\frac{1}{2}(s-t)\right) - \cot\left(\frac{1}{2}(s+t)\right)\right\}\{\tau_n(t_0) - \tau_n(s)\}\,ds, \\
&\geq \frac{1}{3}\int_0^{t_0} K(t,s)\frac{\sin\theta_n(s)}{\nu_n + \int_0^s \sin\theta_n(v)\,dv}\,ds, \qquad \text{for all } t \in (0, t_0),
\end{aligned}$$

where we have used the consequence of (4.2) that $\tau_n(t_0) \geq \tau_n(t)$ for all $t \in (t_0, \pi]$, for all $n \geq 1$. There is no loss of generality to assume that $0 < t_0 \leq \pi/4$, and in this situation, (3.10) shows that, for all $t \in (0, t_0)$,

$$\theta_n(t) \geq \frac{\pi}{12}\int_0^{t_0} k(t,s)\frac{\sin\theta_n(s)}{\nu_n + \int_0^s \sin\theta_n(v)\,dv}\,ds,$$

where k is given by (3.2). Let $t_1 := t_0/2$. Lemma 3.6 applied to $I := (0, t_1)$ yields a constant $m_1 > 0$ such that, for all $n \geq 1$,

$$\nu_n + \int_0^s \sin\theta_n(v)\,dv \geq m_1 s \qquad \text{for all } s \in I.$$

But, for $s > t_1$, hypothesis (4.2) shows that

$$\nu_n + \int_0^s \sin\theta_n(v)\,dv \geq \nu_n + \int_0^{t_1} \sin\theta_n(v)\,dv \geq m_1 t_1 = m_1 s\frac{t_1}{s} \geq \frac{m_1 t_1}{\pi}s,$$

and it follows that there exists $m > 0$ such that

$$\nu_n + \int_0^s \sin\theta_n(v)\,dv \geq m|s| \qquad \text{for all } s \in [-\pi, 0) \cup (0, \pi]. \tag{4.7}$$

For all $n \geq 1$, let $\tilde{\tau}_n := \tau_n - a_n$. It follows from (4.7) that there exists $g \in L^2_{2\pi}$

such that

$$|\tilde{\tau}_n| \le g \quad \text{everywhere on } [-\pi, 0) \cup (0, \pi]. \tag{4.8}$$

In particular, the sequence $\{\tilde{\tau}_n\}_{n\ge1}$ is bounded in $L^2_{2\pi}$, hence the sequence $\{a_n\}_{n\ge1}$ is bounded.

For every $\delta \in (0, \pi]$, using (4.7) in (4.6) yields, for all $n \ge 1$,

$$|\tilde{\tau}_n'(s)| \le \frac{2}{3m\delta} \quad \text{for all } s \in [-\pi, -\delta/2] \cup [\delta/2, \pi].$$

Hence the functions $\{\tilde{\tau}_n\}_{n\ge1}$ are Lipschitz continuous with the same Lipschitz constant on $[-\pi, -\delta/2] \cup [\delta/2, \pi]$, and hence Hölder continuous with the same constant for any exponent $\alpha \in (0, 1)$. The local version of Privalov's Theorem shows that $\{\theta_n\}_{n\ge1}$ are also Hölder continuous with the same constant for any exponent $\alpha \in (0, 1)$ on the interval $[-\pi, -\delta] \cup [\delta, \pi]$.

Since this is true for all $\delta \in (0, \pi]$, the Ascoli-Arzela Theorem and a diagonalization argument show that there are subsequences of $\{\tilde{\tau}_n\}_{n\ge1}$ and $\{\theta_n\}_{n\ge1}$ which converge pointwise on $[-\pi, 0) \cup (0, \pi]$ to functions $\tilde{\tau}$, θ, the convergence being uniform on any interval $[-\pi, -\delta] \cup [\delta, \pi]$, where $\delta \in (0, \pi]$. Clearly $\tilde{\tau}$, θ are continuous on $[-\pi, 0) \cup (0, \pi]$, and θ is an odd function. For convenience, we use for these convergent subsequences the same notation as for the original sequences.

Now (4.7) shows, by passing to the limit as $n \to \infty$, that

$$\int_0^s \sin\theta(v)\, dv \ge m|s| \quad \text{for all } s \in [-\pi, 0) \cup (0, \pi].$$

The definition of $\tilde{\tau}_n$ and (4.1b) show, by passing to the limit as $n \to \infty$, that necessarily

$$\tilde{\tau}(s) = -\frac{1}{3}\log\Big(\int_0^s \sin\theta(v)\, dv\Big) \quad \text{for all } s \in [-\pi, 0) \cup (0, \pi].$$

Now using (4.5) and (4.8), the Dominated Convergence Theorem shows that $\tilde{\tau}_n \to \tilde{\tau}$ and $\theta_n \to \theta$ in $L^2_{2\pi}$ as $n \to \infty$. This implies that there exists $a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$ and, if $\tau := \tilde{\tau} + a$, then τ has zero mean.

Finally note that, since (4.1a) holds and $\mathcal{C}$ is a continuous operator from $L^2_{2\pi}$ to $L^2_{2\pi}$, hence a closed operator, it follows that $\theta = \mathcal{C}\tau$. This completes the proof. □

4.2 Gibbs Phenomenon

Krasovskii [12] conjectured that any smooth solution $\tilde{\theta}$ to Nekrasov's equation satisfies $\max_{t\in[0,\pi]}\tilde{\theta}(t)\le\pi/6$. This was disproved by McLeod [15], who showed that in any sequence of solutions to (4.4) with $\nu_n\to 0$, where θ_n is continuous, odd and $0\le\theta_n\le\pi/2$ on $[0,\pi]$, one has $\max_{t\in[0,\pi]}\theta_n(t)>\pi/6$ for all n sufficiently large. Thus a Gibbs Phenomenon occurs. Here we show that the existence of solutions with $\max_{t\in[0,\pi]}\tilde{\theta}(t)>\pi/6$ is ensured in a more general context.

Theorem 4.2. *Let $\{\theta_n\}_{n\ge1}$ be as in Theorem 4.1. Then $\max_{t\in[0,\pi]}\theta_n(t)>\pi/6$ for all n sufficiently large.*

As in [15], our analysis involves the integral equation

$$\phi^\star(x)=\frac{1}{3}\int_0^\infty k(x,y)\frac{\sin\phi^\star(y)}{1+\int_0^y\sin\phi^\star(u)\,du}\,dy,\qquad x\in(0,\infty),\tag{4.9a}$$

$$\inf_{x\in(0,\infty)}\phi^\star(x)\ge0\quad\textit{and}\quad\sup_{x\in(0,\infty)}\phi^\star(x)\le\pi/2,\tag{4.9b}$$

where k is given by (3.2). This equation, first derived in [15], is sometimes called the 'boundary layer equation'.

Theorem 4.3. *Let $\{\theta_n\}_{n\ge1}$ be as in Theorem 4.1. Let $A_n:=\max_{t\in[0,\pi]}\theta_n(t)$. Let A be any number such that $A_n\le A$ for all $n\ge1$. Then there exists a solution $\phi^\star\in C_b(0,\infty)$ of (4.9) such that $\sup_{x\in(0,\infty)}\phi^\star(x)\le A$.*

For the purposes of displaying the Gibbs Phenomenon, the relevant choice in Theorem 4.3 is $A:=\pi/6$. However, Theorem 4.3 has a more general meaning. If one chooses $A:=\pi/2$, this result actually means, as it will become apparent from the proof, that the sequence of functions $\{\theta_n\}_{n\ge1}$, with their arguments suitably rescaled, converges in a certain sense along a subsequence to a function $\phi^\star$ satisfying (4.9).

Theorem 4.4. *Any solution $\phi^\star$ of (4.9) satisfies $\sup_{x\in(0,\infty)}\phi^\star(x)>\pi/6$.*

The proof of Theorem 4.2 is based on Theorem 4.3 and Theorem 4.4. To prove Theorem 4.2, we argue by contradiction and assume that there exists a subsequence $\{\theta_{n_l}\}_{l\ge1}$ for which $A_{n_l}\le\pi/6$, for all $l\ge1$. Then Theorem 4.3 can be applied to the sequence $\{\theta_{n_l}\}_{l\ge1}$ (rather than to the original sequence $\{\theta_n\}_{n\ge1}$), with $A := \pi/6$, to yield a solution $\phi^\star$ of (4.9) with $\sup_{x\in(0,\infty)} \phi^\star(x) \leq \pi/6$. But this would contradict Theorem 4.4. This would prove Theorem 4.2, once Theorem 4.3 and 4.4 had been proved.

The proofs of Theorem 4.3 and Theorem 4.4 here are simpler than in [15], in that we avoid rather awkward estimates on the derivatives of the functions θ_n, for $n \geq 1$.

In [15], McLeod asked the question of whether solutions of (4.9) take the value $\pi/6$ infinitely often, or only at finitely many points. Since the existence of at least one solution to (4.9) is known, it is therefore of interest to study its properties, with a view to answering McLeod's question. Although we have not achieved this final goal, there follows a new result on this problem.

Theorem 4.5. *Any solution $\phi^\star$ of (4.9) satisfies $\lim_{x\to\infty} \phi^\star(x) = \pi/6$.*

To proof of Theorem 4.5 is based the same idea as that of Theorem 3.9. The key ingredient is the following result.

Theorem 4.6. *Suppose that there exists a sequence $d_n \to \infty$ such that $\phi^\star(d_n) \to a$, where $0 \leq a \leq \pi/2$. Then $a > 0$ and there exists a solution $\phi^\star$ of (3.27), with $\mu = 1/3$, such that $\phi^\star(1) = a$.*

Once Theorem 4.6 is proved, then Theorem 3.12 shows that necessarily $a = \pi/6$. This proves Theorem 4.5.

There follow the proofs of Theorem 4.3, Theorem 4.4 and Theorem 4.6.

Proof of Theorem 4.3. For $n \geq 1$, let $\tilde{\phi}_n : (0,\infty) \to \mathbb{R}$ defined by $\tilde{\phi}_n(x) := \theta_n(2\arctan x)$, for all $x \in (0,\infty)$. It follows from (4.4), using (3.3) and the change of variables $s = 2\arctan y$ in the integral, that, for all $x \in (0,\infty)$,

$$\tilde{\phi}_n(x) = \int_0^\infty k(x,y)\frac{1}{y}\tilde{\rho}_n(y)\,dy, \tag{4.10}$$

where

$$\tilde{\rho}_n(y) := \frac{1}{3}\frac{\sin\theta_n(2\arctan y)}{\nu_n + \int_0^{2\arctan y}\sin\theta_n(v)\,dv}\frac{2y}{1+y^2} \quad \text{for all } y \in (0,\infty).$$

We now claim that the functions $\tilde{\rho}_n$ are uniformly bounded. Indeed, (4.7) shows that

$$|\tilde{\rho}_n(y)| \leq \frac{1}{3}\frac{2y}{1+y^2}\frac{1}{2m_1\arctan y} \leq M, \tag{4.11}$$

where M is independent of $n \geq 1$ and $y \in (0, \infty)$.

Note also that

$$\tilde{\rho}_n(y) = \frac{1}{3} \frac{\sin \tilde{\phi}_n(y)}{\nu_n + \displaystyle\int_0^y \sin \tilde{\phi}_n(v) \frac{2}{1+v^2}\, dv} \frac{2y}{1+y^2}. \tag{4.12}$$

For $n \geq 1$, let $b_n := \nu_n/2$, and the functions $\phi_n, \rho_n : (0,\infty) \to \mathbb{R}$ given by $\phi_n(x) := \tilde{\phi}_n(b_n x)$, $\rho_n(x) := \tilde{\rho}_n(b_n x)$, for all $x \in (0, \infty)$, so that, by a change of variables in (4.10),

$$\phi_n(x) = \int_0^\infty k(x,y) \frac{1}{y} \rho_n(y)\, dy \quad \text{for all } x \in (0,\infty). \tag{4.13}$$

Clearly the functions ρ_n are uniformly bounded by M. By Corollary 3.4, the sequence of functions $\{\phi_n\}_{n\geq 1}$ is uniformly bounded and equicontinuous on $(0,\infty)$. By considering an expanding sequence of compact intervals whose union is $(0,\infty)$, the Ascoli-Arzela Theorem and a diagonalization argument yield a subsequence of $\{\phi_n\}_{n \geq 1}$ which converges, uniformly on compact intervals in $(0, \infty)$, to a function $\phi^\star \in C_b(0,\infty)$. For convenience, we use for this convergent subsequence the same notation as for the original sequence.

Fix $x \in (0,\infty)$. We want to pass to the limit in (4.13) using the Dominated Convergence Theorem. It is clear that

$$\left| k(x,y) \frac{1}{y} \rho_n(y) \right| \leq M k(x,y) \frac{1}{y}, \quad \forall n \geq 1,$$

where M is as in (4.11). Since, by (3.6),

$$\int_0^\infty k(x,y) \frac{1}{y}\, dy < \infty,$$

it remains to examine the pointwise convergence of the integrands. For each $y \in (0,\infty)$ and $n \geq 1$, we have

$$\rho_n(y) = \frac{1}{3} \frac{\sin \phi_n(y)}{1 + \displaystyle\int_0^y \sin \phi_n(v) \frac{1}{1+b_n^2 v^2}\, dv} \frac{y}{1+b_n^2 y^2}.$$

Since $b_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \rho_n(y) = \frac{1}{3}\frac{y\sin\phi^\star(y)}{1+\int_0^y \sin\phi^\star(u)\,du} \qquad \text{for all } y \in (0,\infty).$$

Hence, passing to the limit in (4.13) yields that $\phi^\star$ satisfies (4.9a). Since, for any $x \in (0,\infty)$,

$$\phi^\star(x) := \lim_{n\to\infty} \tilde{\phi}_n(b_n x) = \lim_{n\to\infty} \theta_n(2\arctan(b_n x)), \tag{4.14}$$

it is obvious that (4.9b) also holds. Finally, if $\{A_n\}_{n\geq 1}$ and A are as in the statement of the Theorem, (4.14) shows that $\sup_{x\in(0,\infty)} \phi^\star(x) \leq A$, as required. □

Proof of Theorem 4.4. Let $\phi^\star$ satisfy (4.9). Lemma 3.6, with $I := (0,\infty)$, shows that there exists $m > 0$ such that

$$1 + \int_0^x \sin\phi^\star(v)\,dv \geq mx \qquad \text{for all } x \in (0,\infty). \tag{4.15}$$

As in [15], we deduce that, for all z_1, $z_2 \in (0,\infty)$ with $z_1 < z_2$,

$$\begin{aligned}&\int_{z_1}^{z_2} \frac{\phi^\star(u) - \pi/6}{u}\,du \\ &= \frac{1}{3}\int_0^\infty k(1,v)\frac{1}{v}\left\{\log\left(\frac{1+\int_0^{z_2 v}\sin\phi^\star(u)du}{z_2 v}\right) - \log\left(\frac{1+\int_0^{z_1 v}\sin\phi^\star(u)du}{z_1 v}\right)\right\}dv.\end{aligned} \tag{4.16}$$

Since (4.15) holds, the right-hand side of (4.16) is bounded by a constant independent of z_1, z_2, for $1 \leq z_1 < z_2 < \infty$. Hence the left hand-side is similarly bounded.

Suppose now for a contradiction that $\phi^\star(x) \leq \pi/6$ for all $x \in (0,\infty)$. In this situation, it follows from (4.16) that necessarily

$$0 \leq \int_1^\infty \frac{\pi/6 - \phi^\star(u)}{u}\,du < +\infty. \tag{4.17}$$

We claim that this implies that $\lim_{x\to\infty} \phi^\star(x) = \pi/6$. Indeed, suppose that this is not so, and let $\delta > 0$ and $\{c_n\}_{n\geq 1}$ be a sequence such that $\phi^\star(c_n) \leq \pi/6 - 2\delta$, for all $n \geq 1$. It can be assumed with no loss of generality that $c_{n+1}/c_n \geq a$, for some $a > 1$. Since (4.15) holds, it follows from (4.9) that

$$\phi^\star(x) = \int_0^\infty k(x,y)\frac{1}{y}\rho^\star(y)\,dy \quad \text{for all } x \in (0,\infty),$$

where $\rho^\star$ is a bounded function. The estimates in Lemma 3.2 show that there exists b with $1 < b < a$, such that $|\phi^\star(x) - \phi^\star(x_0)| \leq \delta$ for all x, $x_0 \in (0,\infty)$ such that $1 \leq x/x_0 \leq b$. This implies that $\phi^\star(u) \leq \pi/6 - \delta$ if $c_n \leq u \leq bc_n$, and therefore

$$\int_{c_n}^{bc_n} \frac{\pi/6 - \phi^\star(u)}{u}\,du \geq \delta \int_{c_n}^{bc_n} \frac{1}{u}\,du \geq \delta \log b \quad \text{for all } n \geq 1. \tag{4.18}$$

Since the intervals (c_n, bc_n), with $n \geq 1$, are all disjoint, (4.18) contradicts (4.17).

This proves that, if $\phi^\star(x) \leq \pi/6$ for all $x \in (0,\infty)$, then $\lim_{x\to\infty} \phi^\star(x) = \pi/6$. Whilst this was previously known, the above proof circumvents the estimates in [15] on the derivative of $\phi^\star$. The rest of the proof can be carried out as in [15], where no further use is made of the estimates on derivatives. □

It is easy to see that a slight modification of the above argument yields the conclusion that $\lim_{x\to\infty} \phi^\star(x) = \pi/6$ for any solution $\phi^\star$ of (4.9) having the property that $\phi^\star(x) - \pi/6$ does not change sign for x sufficiently large. It does not seem to yield the full result of Theorem 4.4.

Proof of Theorem 4.6. For $n \geq 1$, let $e_n := 1/d_n$, and let $\phi_n^\star : (0,\infty) \to \mathbb{R}$ be given by $\phi_n^\star(x) = \phi^\star(d_n x)$, for all $x \in (0,\infty)$. Since (4.15) holds, Corollary 3.4 shows that the sequence $\{\phi_n^\star\}_{n\geq 1}$ is uniformly bounded and equicontinuous on $(0,\infty)$. The Ascoli-Arzela Theorem shows that there is a subsequence which converges uniformly on compact intervals to a function $\phi^\star \in C_b(0,\infty)$. It is immediate from (4.15) that

$$e_n + \int_0^x \sin\phi_n^\star(v)\,dv \geq mx \quad \text{for all } x \in (0,\infty),$$

and hence, passing to the limit as $n \to \infty$ yields, since $e_n \to 0$, that

$$\int_0^x \sin\phi^\star(v)\,dv \geq mx \quad \text{for all } x \in (0,\infty).$$

Since, for all $n \geq 1$ and all $y \in (0, \infty)$,

$$\phi_n^\star(x) = \frac{1}{3} \int_0^\infty k(x,y) \frac{1}{y} \frac{y \sin \phi_n^\star(y)}{e_n + \int_0^y \sin \phi_n^\star(u)\, du}\, dy,$$

the Dominated Convergence Theorem yields that ϕ^* satisfies (3.27a) with $\mu := 1/3$. Obviously $\sup_{x \in (0,\infty)} \phi^*(x) \leq \pi/2$ and, by Lemma 3.6, $\inf_{x \in (0,\infty)} \phi^*(x) > 0$. This shows that (3.27b) holds, and that $a > 0$. Clearly $\phi^*(1) = a$. This completes the proof. $\square$

Chapter 5

Analytic Continuation in the Complex Domain

By extending the method in [20], we show that, for a general class of Bernoulli problems, a certain holomorphic function originally defined in a semi-infinite strip has an analytic continuation to the whole strip. As a consequence of this approach, we derive an interesting property of the Fourier coefficients of the function which gives the angle between the tangent to the free boundary and the horizontal. The method, which was also used in [19] in the proof of the first Stokes conjecture, is to study the solution of an ordinary differential equation in the complex domain. Arguments of this type have also played an important role in the proof [21] of the second Stokes conjecture, on the convexity of the profile of a wave of extreme form.

5.1 A Class of Bernoulli Free Boundaries

Consider a free boundary of the form $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous, even, with minimal period 2π and exactly one local maximum per period, located at $X = 0 \mod 2\pi$. We assume that λ is real-analytic on $(-\infty, 0)$ and

$$\lambda \text{ is decreasing on } (-\infty, 0]. \tag{5.1}$$

Let $Y_c := \eta(0)$ and $Y_t := \eta(\pi)$. The regularity results in [27] show that, if $Y_c < 0$, then $\mathcal{S}$ is a real-analytic curve, while, if $Y_c = 0$, then $\mathcal{S}_{\mathcal{N}} = \{(2n\pi, 0) : n \in \mathbb{Z}\}$

and $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$ is a union of real-analytic curves. Depending on whether $Y_c < 0$ or $Y_c = 0$, it will be assumed that the function η is real-analytic either on $\mathbb{R}$, or on $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$. We also assume that

$$\eta'(X) < 0 \quad \text{for all } X \in (0, \pi). \tag{5.2}$$

Let $g : [0, \infty) \to [0, \infty)$ be given by

$$g(r) = \lambda(-r) \quad \text{for all } r \in [0, \infty). \tag{5.3}$$

The case $g(r) = cr$, where $c > 0$, is the one arising in the Stokes waves problem as discussed in [20], while the case $g(r) = cr^{\alpha-1}$, where $c > 0$ and $\alpha \geq 2$, was considered in [21].

As noted in Chapter 1, if ψ satisfies (1.1) and φ is a harmonic conjugate of $-\psi$ in Ω, then the analytic function $\omega := \varphi + i\psi$ is a conformal bijection from Ω onto the open lower half-plane $\mathbb{C}_-$. In this chapter only, the mapping ω will be denoted instead by $\tilde{z} := \tilde{x} + i\tilde{y}$, and its inverse by $\widetilde{Z} := \widetilde{X} + i\widetilde{Y}$. We continue to identify the complex planes of generic variables $Z = X + iY$ and $z = x + iy$ with the real planes of variables (X, Y) and (x, y), respectively.

Let us introduce some notation. For every $A, B \in \mathbb{R}$ with $A < B$, consider the sets in the (X, Y)-plane

$$\Omega_{A,B} := \{(X, Y) \in \Omega : A < X < B\},$$

and

$$\mathcal{S}_{A,B} := \{(X, \eta(X)) : A < X < B\},$$

and, for $a, b \in \mathbb{R}$ with $a < b$, the sets in the (x, y)-plane

$$\Pi_{a,b} := \{(x, y) : a < x < b, y \in \mathbb{R}\} \quad \text{and} \quad \Pi^{\pm}_{a,b} := \Pi_{a,b} \cap \mathbb{R}^2_{\pm}.$$

Then $\tilde{z} = \tilde{x} + i\tilde{y}$ is a bijection from $\Omega_{0,\pi}$ onto $\Pi^-_{0,\pi}$, and from $\Omega_{\pi,2\pi}$ onto $\Pi^-_{\pi,2\pi}$. Let U be the analytic function on $\mathbb{C}_-$ given by

$$U(z) = \frac{d\tilde{z}}{dZ}(\widetilde{Z}(z)) \quad \text{for all } z \in \mathbb{C}_-, \tag{5.4}$$

and let p and θ be harmonic functions on $\mathbb{R}^2_-$ given by

$$U(x+iy) = e^{p(x,y) + i\theta(x,y)} \quad \text{for all } (x,y) \in \mathbb{R}^2_-. \tag{5.5}$$

The properties of the function $\theta(x,y)$ for $x \in (0, 2\pi)$, $y \in (-\infty, 0]$, are of particular interest, since this function gives the angle between the tangent to the level sets of ψ, including the free boundary, and the horizontal. We study the possibility that θ has an extension as a harmonic function in the strip $\Pi_{0,2\pi}$.

We start by collecting properties of the partial derivatives of ψ in Ω, since these will be of importance later.

Since $\psi = 0$ on $\mathcal{S}$ and $\psi < 0$ in Ω, it follows by Hopf Boundary-Point Lemma that $\psi_Y > 0$ on $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$. Therefore, by the Maximum Principle,

$$\psi_Y > 0 \quad \text{in } \Omega. \tag{5.6}$$

Since ψ is 2π-periodic and even, it follows that

$$\psi_X(n\pi, Y) = 0 \quad \text{for all } n \in \mathbb{Z}, Y < \eta(n\pi), \tag{5.7}$$

and hence

$$\psi_{XY}(n\pi, Y) = \psi_{XYY}(n\pi, Y) = 0 \quad \text{for all } n \in \mathbb{Z}, Y < \eta(n\pi). \tag{5.8}$$

Differentiating $\psi(X, \eta(X)) = 0$ with respect to X where possible yields

$$\psi_X + \psi_Y \eta' = 0, \tag{5.9}$$

from where we conclude that $\psi_X > 0$ in $\mathcal{S}_{0,\pi}$ and therefore, by (5.7) and the Maximum Principle,

$$\psi_X > 0 \quad \text{in } \Omega_{0,\pi}. \tag{5.10}$$

It follows from (5.7) and (5.10), using Hopf Boundary-Point Lemma, that

$$\psi_{XX}(\pi, Y) < 0 \quad \text{for all } Y < Y_t, \tag{5.11a}$$

$$\psi_{XX}(0, Y) > 0 \quad \text{for all } Y < Y_c. \tag{5.11b}$$

Hence $\psi_{XX}(\pi, Y_t) \leq 0$ and $\psi_{XX}(0, Y_c) \geq 0$ if $Y_c < 0$. We now aim to prove that in fact $\psi_{XX}(\pi, Y_t) < 0$ and $\psi_{XX}(0, Y_c) > 0$ if $Y_c < 0$, since these estimates will be crucial later on. Since, by (5.7) and (5.1),

$$\psi_Y^2(\pi, Y_t) = \lambda(Y_t) \geq \lambda(\eta(X)) \geq \psi_Y^2(X, \eta(X)) \quad \text{for all } X \in \mathbb{R},$$

it follows that the harmonic function ψ_Y has a maximum in $\overline{\Omega}$ at (π, Y_t), and hence by Hopf Boundary-Point Lemma, $\psi_{YY}(\pi, Y_t) > 0$. Therefore, since ψ is a harmonic function,

$$\psi_{XX}(\pi, Y_t) < 0. \tag{5.12}$$

The preceding argument cannot be repeated to show that $\psi_{XX}(\pi, Y_c) > 0$ when $Y_c < 0$. We now prove this by an application of Serrin Corner-Point Lemma. We argue by contradiction and assume that $\psi_{XX}(0, Y_c) = 0$. Under this assumption, differentiating once more with respect to X in (5.9), and evaluating the result at $X = 0$ yields, since $\eta'(0) = 0$, that $\eta''(0) = 0$. Differentiating now with respect to X the relation $|\nabla\psi|^2(X, \eta(X)) - \lambda(\eta(X)) = 0$ leads to

$$2\psi_X(\psi_{XX} + \psi_{XY}\eta') + 2\psi_Y(\psi_{XY} + \psi_{YY}\eta') - \lambda'(\eta)\eta' = 0.$$

Differentiating once more in the above at $X = 0$ leads to, where the terms containing η' and η'' are not written down,

$$2\psi_{XX}\psi_{XX} + 2\psi_X\psi_{XXX} + 2\psi_{YX}\psi_{XY} + 2\psi_Y\psi_{XYX} = 0 \quad \text{at } (0, Y_c).$$

This shows that $\psi_{XXY}(0, Y_c) = 0$. Together with $\psi_{XX}(0, Y_c) = 0$ and (5.8), this shows that all the first and second order partial derivatives of the function ψ_X are zero at $(0, Y_c)$. But this contradicts Lemma 2.21, since ψ_X is harmonic in $\Omega_{0,\pi}$ and has a minimum in $\overline{\Omega_{0,\pi}}$ at $(0, Y_c)$. We conclude that necessarily

$$\psi_{XX}(0, Y_c) > 0 \quad \text{if } Y_c < 0. \tag{5.13}$$

This argument based on Serrin's Lemma would as well apply to give an alternative proof of (5.12), which would not require that λ satisfies (5.1).

It follows from (5.4)-(5.7) and (5.10) that

$$0 < \theta < \pi/2 \quad \text{in } \Pi^-_{0,\pi}, \tag{5.14}$$

and

$$\theta(0,y) = \theta(\pi,y) = 0 \quad \text{for } -\infty < y < 0.$$

Let σ be defined by $\sigma(x) := -\widetilde{Y}(x+i0)$, for all $x \in (0,2\pi)$. Then (1.1j) shows that

$$e^{2p(x)} = g(\sigma(x)) \quad \text{for all } x \in (0,2\pi).$$

Since, as a consequence of (5.4)-(5.5), $e^{-p-i\theta} = d\widetilde{Z}/dz$ on $\mathbb{C}_-$, it follows that

$$\frac{d\sigma}{dx}(x) = e^{-p(x)}\sin\theta(x) = \frac{1}{2i}\left(\frac{e^{p(x)+i\theta(x)}}{e^{2p(x)}} - e^{-p(x)-i\theta(x)}\right), \quad x \in (0,2\pi),$$

from where we deduce that

$$\frac{d\sigma}{dx}(x) = \frac{1}{2i}\left(\frac{U(x)}{g(\sigma(x))} - \frac{1}{U(x)}\right) \quad \text{for all } x \in (0,2\pi). \tag{5.15}$$

Since g was assumed real-analytic on $(0,\infty)$, it follows that it has a holomorphic extension to a neighbourhood Ξ in the complex plane of the interval $(0,\infty)$ of the real line, such that $g \neq 0$ on Ξ.

Consider now the following ODE in the complex domain

$$\frac{d\sigma}{dz}(z) = \frac{1}{2i}\left(\frac{U(z)}{g(\sigma(z))} - \frac{1}{U(z)}\right), \tag{5.16a}$$

$$\sigma(x+i0) = -\widetilde{Y}(x+i0) \quad \text{for } x \in (0,2\pi). \tag{5.16b}$$

In (5.16a), the function U is a known holomorphic function in $\Pi^-_{0,2\pi} \cup \Upsilon$, where Υ is a neighbourhood of the interval $(0,2\pi)$ of the real line. Since (5.15) holds, Cauchy's Theorem in the theory of ODE in the complex domain shows that (5.16) has an unique holomorphic solution σ on a neighbourhood $\widehat{\Upsilon}$ of the interval $(0,2\pi)$ of the real axis, where $\widehat{\Upsilon} \subseteq \Upsilon$ and $\sigma(\widehat{\Upsilon}) \subseteq \Xi$. To ensure that this solution can be analytically continued to $\Pi^-_{0,2\pi}$, we make further assumptions on g.

Suppose that g is holomorphic in $\mathbb{C} \setminus (-\infty,0]$, continuous in $\mathbb{C} \setminus (-\infty,0)$ with $g(0) = 0$, and real-valued on $(0,\infty)$. Let G denote a primitive of g with

$G(0) = 0$. We further assume that there exists an even continuous function $\gamma : \mathbb{R} \to \mathbb{R}$, which is real-analytic on $\mathbb{R} \setminus \{0\}$, such that, if

$$\Gamma := \{\gamma(\sigma_2) + i\sigma_2 : \sigma_2 \in \mathbb{R}\}, \qquad \Gamma^{\pm} := \Gamma \cap \mathbb{C}_{\pm}, \tag{5.17}$$

and

$$\Theta := \{\sigma_1 + i\sigma_2 : \sigma_2 \in \mathbb{R},\, \sigma_1 > \gamma(\sigma_2)\}, \qquad \Theta^{\pm} := \Theta \cap \mathbb{C}_{\pm}, \tag{5.18}$$

then

$$\pm\ \mathrm{Im}\ G > 0 \quad \text{in } \Theta^{\pm}, \tag{5.19a}$$

$$\mathrm{Im}\ G = 0 \quad \text{on } \Gamma, \tag{5.19b}$$

and there exist $c, C, D > 0$ such that

$$|g(\sigma)| \geq c \quad \text{in} \quad \Theta \cap \{\sigma : |\sigma| \geq C\}, \tag{5.19c}$$

and

$$\pm\ \mathrm{Im}\ g > 0 \quad \text{in} \quad \Theta^{\pm} \cap \{\sigma : |\sigma| \leq D\}. \tag{5.19d}$$

A possible geometry of the domain Θ is illustrated in Figure 5-1.

Note that, for the cases treated in [20] and [21], where $G(\sigma) = C\sigma^{\alpha}$ with $C > 0$ and $\alpha \geq 2$, the set Θ is precisely the angular sector $\{\sigma \in \mathbb{C} : -\pi/\alpha < \arg \sigma < \pi/\alpha\}$ and (5.19) is trivially satisfied.

We now derive some further properties of g and G in Θ. It follows from (5.19a), (5.19b) and Hopf Boundary-Point Lemma that

$$\pm\ \mathrm{Im}\ g > 0 \quad \text{on } \Gamma^{\pm}. \tag{5.20}$$

Lemma 1.6 shows that G is a conformal bijection from Θ^- onto the open lower half-plane, which in addition maps ∞ onto ∞, and extends as a homeomorphism from the closure of Θ^- onto the closed lower half-plane. By symmetry arguments, G is also a conformal bijection from Θ^+ onto the open upper half-plane, which maps ∞ onto ∞, and extends as a homeomorphism from the closure of Θ^+ onto the closed upper half-plane. As a consequence, $g \neq 0$ in Θ. Moreover, we can conclude from (5.20) that $g \neq 0$ in $\overline{\Theta} \setminus \{0\}$.

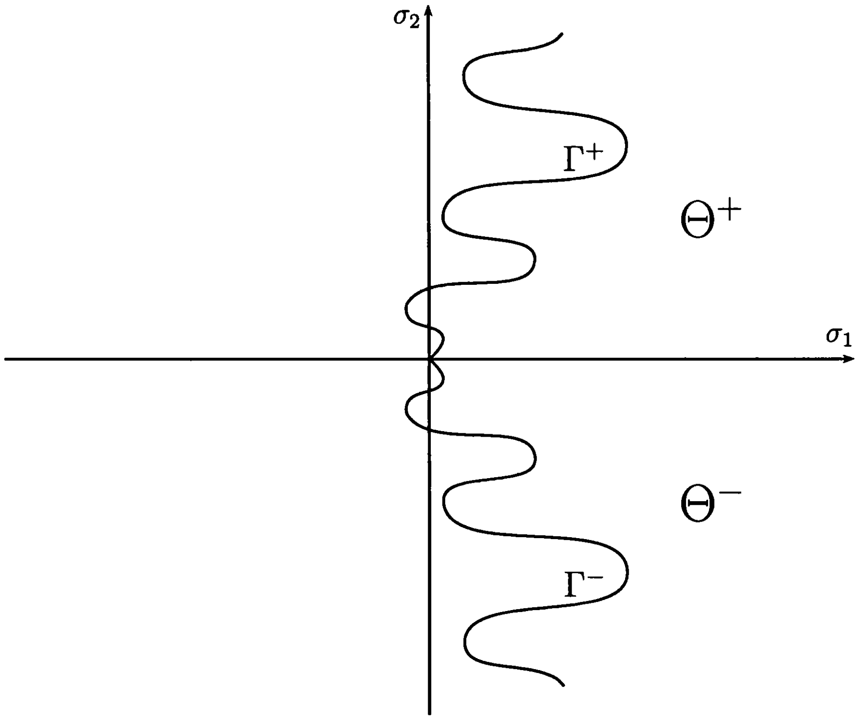

Figure 5-1: A domain Θ and its boundary Γ

Consider now the harmonic function $\operatorname{Im} 1/g$ which, by (5.19c), is bounded in $\Theta^- \cap \{\sigma : |\sigma| \geq D/2\}$. Using (5.19c), (5.19d) and the fact that g is real on the positive real axis, the Maximum Principle yields that

$$\operatorname{Im} g < 0 \quad \text{in} \quad \Theta^- \cap \{\sigma : |\sigma| \geq D/2\}.$$

Therefore, taking also into account (5.19d), we conclude that

$$\operatorname{Im} g < 0 \quad \text{in} \quad \Theta^-. \tag{5.21}$$

A similar argument yields $\operatorname{Im} g > 0$ in Θ^+. It follows from Hopf Boundary-Point Lemma that, if $g := g_1 + ig_2$, then $g_{2,y} > 0$ on the interval $(0, \infty)$ of the real axis, and therefore, by the Cauchy-Riemann equations, $g_{1,x} = g' > 0$ on $(0, \infty)$, a fact which is in agreement with (5.1).

5.2 Complex ODE

Since $g \neq 0$ in Θ, it follows that (5.16a) can be written in the form

$$\frac{d\sigma}{dz}(z) = \mathcal{F}(z,\sigma),$$

where $\mathcal{F} : (\Pi^-_{0,2\pi} \cup \Upsilon) \times \Theta \to \mathbb{C}$ given by

$$\mathcal{F}(z,\sigma) := \frac{1}{2i}\left(\frac{U(z)}{g(\sigma)} - \frac{1}{U(z)}\right) \quad \text{in} \quad (\Pi^-_{0,2\pi} \cup \Upsilon) \times \Theta,$$

is a holomorphic function with respect to both its arguments.

Theorem 5.1. *Under the assumptions* (5.19), *the Cauchy problem* (5.16) *has a unique holomorphic solution* σ *in* $\Pi^-_{0,2\pi}$. *Moreover,*

$$\sigma(z) \in \Theta^- \quad \text{for all } z \in \Pi^-_{0,\pi}, \qquad \sigma(z) \in \Theta^+ \quad \text{for all } z \in \Pi^-_{\pi,2\pi}, \tag{5.22a}$$

$$\sigma(\pi + iy) \in (0,\infty) \quad \text{for all} \quad y \in (-\infty, 0), \tag{5.22b}$$

and, for any z_1, $z_2 \in \Pi^-_{0,2\pi}$ *which are symmetric with respect to the line* $x = \pi$, *the values* $\sigma(z_1)$ *and* $\sigma(z_2)$ *are symmetric with respect to the real axis.*

In addition, σ *has a continuous extension to the closure of* $\Pi^-_{0,2\pi}$. *Let*

$$\mathcal{Z} := \{y \in (-\infty, 0] : \sigma(0 + iy) = 0\}.$$

Then there exists $y_\star \leq 0$ *such that* $\mathcal{Z} = \{y_\star\}$, *and* $\sigma(iy) \in \Theta^-$ *for all* $y \in (-\infty, y_\star)$. *If* $Y_c < 0$ *then* $y_\star < 0$ *and* $\sigma(iy)$ *is real and positive on* $(y_\star, 0]$. *If* $Y_c = 0$ *then* $y_\star = 0$. *We write*

$$\tilde{\phi}(y) := -\arg g(\sigma(iy)) \quad \text{for all} \quad y \in (-\infty, 0] \setminus \{y_\star\}. \tag{5.23}$$

Then $0 < \tilde{\phi} < \pi$ *everywhere on* $(-\infty, y_\star)$, *and* $\tilde{\phi} = 0$ *on* $(y_\star, 0)$ *if* $y_\star < 0$.

Proof of Theorem 5.1. We consider the solution, a holomorphic function σ on a neighbourhood $\widehat{\Upsilon}$ of the interval $(0, 2\pi)$ of the real line, of (5.16) given by Cauchy's Theorem. Let $\sigma := \sigma_1 + i\sigma_2$. Since $\sigma_{2,y} = \sigma_{1,x} > 0$ on $(0,\pi)$, there exists a neighbourhood $\widehat{\Upsilon}_1 \subseteq \widehat{\Upsilon}$ of the interval $(0,\pi)$ such that $\sigma(z) \in \Theta^-$ for all $z \in \widehat{\Upsilon}_1 \cap \Pi^-_{0,\pi}$. Similarly there exists a neighbourhood $\widehat{\Upsilon}_2 \subseteq \widehat{\Upsilon}$ of the interval

$(\pi, 2\pi)$ such that $\sigma(z) \in \Theta^+$ for all $z \in \widehat{\Upsilon}_2 \cap \Pi^-_{\pi,2\pi}$. Since $U(\pi+iy) \in \mathbb{R}$ for $y \leq 0$, it follows that $\sigma(\pi + iy) \in (0, \infty)$ for $y \in (-\varepsilon, 0]$, for some $\varepsilon > 0$.

Next, by working in the Z-plane, we show that σ can be analytically continued to $\Pi^-_{0,2\pi}$, and in such a way that (5.22) holds. Note that problem (5.16) is equivalent to the following in the Z-plane,

$$\frac{d\sigma^*}{dZ}(Z) = \frac{1}{2i}\left(\frac{U(\tilde{z}(Z))^2}{g(\sigma^*(Z))} - 1\right), \tag{5.24a}$$

$$\sigma^*(X + i\eta(X)) = -\eta(X) \quad \text{for } X \in (0, \pi), \tag{5.24b}$$

where $\sigma^*(Z) := \sigma(\tilde{z}(Z))$. We have already seen that (5.24) has a solution σ^* in an open set containing $\mathcal{S}_{0,2\pi}$, and there exist $\epsilon > 0$ and neighbourhoods $\widehat{\Sigma}_1$ of $\mathcal{S}_{0,\pi}$ and $\widehat{\Sigma}_2$ of $\mathcal{S}_{\pi,2\pi}$, such that

$$\sigma^* \in \Theta^- \text{ on } \widehat{\Sigma}_1 \cap \Omega_{0,\pi}, \qquad \sigma^* \in \Theta^+ \text{ on } \widehat{\Sigma}_2 \cap \Omega_{\pi,2\pi}$$
$$\sigma^*(\pi + iY) \in (0, \infty) \text{ for } Y \in (Y_t - \epsilon, Y_t].$$

For every $(X_0, Y_0) \in \mathcal{S}_{0,2\pi}$, the ray $\{(X_0, Y) : -\infty < Y < Y_0\}$ is contained in $\Omega_{0,2\pi}$, and the union of all such rays covers $\Omega_{0,2\pi}$. It suffices to show that σ^* can be analytically continued along each of these rays, and that the continuation satisfies

$$\sigma^* \in \Theta^- \text{ on } \Omega_{0,\pi}, \qquad \sigma^* \in \Theta^+ \text{ on } \Omega_{\pi,2\pi} \tag{5.25a}$$

$$\sigma^*(\pi + iY) \in (0, \infty) \text{ for all } Y \in (-\infty, Y_t]. \tag{5.25b}$$

Along such a ray, (5.24) leads to the ordinary differential equation

$$\frac{d\sigma^*}{dY} = \frac{1}{2}\left(\frac{U(\tilde{z})^2}{g(\sigma^*)} - 1\right), \quad Y < Y_0, \tag{5.26}$$

with

$$\sigma^*(X_0 + iY_0) = -Y_0.$$

Consider first the case $X_0 \in (0, \pi)$. Let (Y_1, Y_0) be the maximal interval of existence of an analytic solution σ^* of (5.26) with $\sigma^* \in \Theta^-$. We want to show that $Y_1 = -\infty$. First we claim that, for all $Y \in (Y_1, Y_0)$,

$$|\sigma^*(X_0 + iY)| \leq C + \frac{1}{2}\left(1 + \frac{M^2}{c}\right)|Y - Y_0|, \tag{5.27}$$

where M is such that $|U| \leq M$ in Ω, the constants c and C are as in (5.19c), and it is assumed with no loss of generality that $|Y_t| < C$.

To prove the claim, fix $Y \in (Y_1, Y_0)$. Then, if $|\sigma^*(X_0 + iY)| > C$, it follows, since $|\sigma^*(X_0 + iY_0)| = |Y_0| \leq |Y_t| < C$, that there exists $Y_2 \in (Y, Y_0)$ such that $|\sigma^*(X_0 + iY_2)| = C$ and

$$|\sigma^*(X_0 + iT)| \geq C \quad \text{for all} \quad T \in (Y, Y_2),$$

and hence, by (5.19c),

$$|g(\sigma^*(X_0 + iT))| \geq c \quad \text{for all} \quad T \in (Y, Y_2).$$

Therefore, from (5.26),

$$\begin{aligned} |\sigma^*(X_0 + iY)| &\leq |\sigma^*(X_0 + iY_2)| + \int_Y^{Y_2} |\sigma^{*\prime}(X_0 + iT)|\, dT \\ &\leq C + \frac{1}{2}\left(1 + \frac{M^2}{c}\right)|Y - Y_2|, \end{aligned}$$

which proves the claim that (5.27) holds.

Multiplying both sides of (5.26) by $g(\sigma^*)$ leads, for $Y_1 < Y < Y_0$, to

$$\frac{d}{dY}(G(\sigma^*)) = \frac{1}{2}(U(\tilde{z})^2 - g(\sigma^*)). \tag{5.28}$$

Suppose now that $Y_1 \neq -\infty$. It follows from (5.27) that σ^* is bounded on (Y_1, Y_0), and so the same is true for $g(\sigma^*)$. Hence (5.28) shows that the mapping $Y \mapsto G(\sigma^*(X_0 + iY))$ is Lipschitz on (Y_1, Y_0), and therefore it has a finite limit, which belongs to the closed lower half-plane, as $Y \searrow Y_1$. Since G is a homeomorphism from $\overline{\Theta^-}$ onto $\mathbb{C}_- \cup \mathbb{R}$, it follows that there exists $\widehat{\sigma^*} := \lim_{Y \searrow Y_1} \sigma^*(X_0 + iY)$ in $\overline{\Theta^-}$. It is clear from the definition of Y_1 that necessarily $\widehat{\sigma^*}$ belongs to the boundary of Θ^-.

It now follows from (5.28) that

$$\frac{d}{dY}(\text{Im } G(\sigma^*)) = \frac{1}{2}(\text{Im } U(\tilde{z})^2 - \text{Im } g(\sigma^*)), \qquad Y_1 < Y < Y_0. \tag{5.29}$$

Note that Im $U^2(\tilde{z}) > 0$ in $\Omega_{0,\pi}$, as a consequence of (5.5) and (5.14). It follows from this and (5.21) that the right-hand side of (5.29) is strictly positive on the interval (Y_1, Y_0). Hence $Y \mapsto \text{Im } G(\sigma^*(X_0 + iY))$ is a strictly increasing function on the interval (Y_1, Y_0). On the other hand, by (5.19a)-(5.19b),

$$\text{Im } G(\sigma^*(X_0 + iY_0)) = 0 = \lim_{Y \searrow Y_1} \{\text{Im } G(\sigma^*(X_0 + iY))\},$$

since $\sigma^*(X_0 + iY) \to \widehat{\sigma^*} \in \partial\Theta^-$ as $Y \searrow Y_1$. We have thus obtained a contradiction. This shows that necessarily $Y_1 = -\infty$.

The case $X_0 \in (\pi, 2\pi)$ can be treated in a very similar way.

Consider now the case when $X_0 = \pi$, and recall that $Y_t := \eta(\pi)$. Let (Y_1, Y_t) be the maximal interval of existence of a real-valued positive solution of (5.26). Suppose for a contradiction that $Y_1 \neq -\infty$. As in the previous part of the proof, we get that σ^* is bounded on (Y_1, Y_t), and there exists $\widehat{\sigma^*} := \lim_{Y \searrow Y_1} \sigma^*(\pi + iY)$. Then necessarily $\widehat{\sigma^*} = 0$. We use the $'$ symbol to denote differentiation with respect to the Y variable (however in (5.30) below the notation g' refers to the ordinary derivative of the function g). Now (5.26) gives that, for $X = \pi$,

$$\sigma^{*\,\prime\prime} = \frac{U(\tilde{z})\{U(\tilde{z})\}'}{g(\sigma^*)} - \frac{U^2(\tilde{z})g'(\sigma^*)\sigma^{*\,\prime}}{2g^2(\sigma^*)}, \tag{5.30}$$

and $\sigma^{*\,\prime}(\pi + iY_t) = 0$. It is a consequence of (5.11a), (5.12) and the Cauchy-Riemann equations that

$$\{U(\tilde{z}(\pi + iY))\}' > 0 \quad \text{for all} \quad Y \leq Y_t. \tag{5.31}$$

It follows that $\sigma^{*\,\prime\prime}(\pi + iY_t) > 0$, and hence $\sigma^{*\,\prime}(\pi + iY) < 0$ for all $Y \in (Y_t - \varepsilon, Y_t)$ for some $\varepsilon > 0$. We claim that $\sigma^{*\,\prime}(\pi + iY) < 0$ for all $Y \in (Y_1, Y_t)$. Indeed, suppose that this does not happen, so that there exists Y_2 with $\sigma^{*\,\prime}(\pi + iY) < 0$ for all $Y \in (Y_2, Y_t)$ and $\sigma^{*\,\prime}(\pi + iY_2) = 0$. This shows that $\sigma^{*\,\prime\prime}(\pi + iY_2) \leq 0$. On the other hand, it follows from (5.30) using (5.31) that $\sigma^{*\,\prime\prime}(\pi + iY_2) > 0$, and we have thus obtained a contradiction. This proves the claim. Hence $\sigma^*(\pi + iY)$

is a decreasing function on $(Y_1, Y_t]$, and so it is impossible that $\lim_{Y \searrow Y_1} \sigma^*(\pi + iY) = 0$. Therefore $Y_1 = -\infty$. Moreover, the preceding considerations show that $\sigma^{*\prime}(\pi + iY) < 0$ for all $Y \in (-\infty, Y_t)$ and, since $g' > 0$ on $(0, \infty)$, it follows from (5.30) that $\sigma^*(\pi + iY)$ is a convex function on $(-\infty, Y_t)$, and therefore necessarily $\sigma^*(\pi + iY) \to \infty$ as $Y \to -\infty$.

He have thus proved the global existence in $\Omega_{0,2\pi}$ of a solution of (5.24) satisfying (5.25), and hence the global existence in $\Pi^-_{0,2\pi}$ of a solution σ of (5.16) satisfying (5.22). Since σ is real on the line $x = \pi$, the symmetry property claimed for σ follows from the Reflection Principle.

Let $\widehat{Y} < Y_t$ and let $\widehat{\Omega} := \{(X, Y) \in \Omega : 0 < X < \pi, \widehat{Y} \le Y\}$. The estimate (5.27) shows that σ^* is uniformly bounded in $\widehat{\Omega}$, and therefore the same is true for $g(\sigma^*)$. Using this fact in (5.28) yields that the complex derivative of the holomorphic function $G(\sigma^*)$ is bounded in $\widehat{\Omega}$. Hence $Z \mapsto G(\sigma^*(Z))$ is a Lipschitz function. Let G^{-1} denote the inverse of the function $G : \Theta^- \to \mathbb{C}_-$. Then G^{-1} is a uniformly continuous function on bounded sets of the closed lower half-plane. It follows that σ^* is a uniformly continuous function on $\widehat{\Omega}$, and so it has a continuous extension to the boundary of this set. Since $\widehat{Y}$ was arbitrary, we have thus obtained the existence of a continuous extension of σ^* to the boundary of $\Omega_{0,\pi}$. The existence of a continuous extension to the boundary of $\Omega_{\pi,2\pi}$ follows from the symmetry of σ^*. It is now clear that σ has a continuous extension to the closure of $\Pi^-_{0,2\pi}$.

For every $X \in (0, \pi)$ and $Y \le \eta(X)$, (5.28) implies that

$$G(\sigma^*(X + i\eta(X))) = G(\sigma^*(X + iY)) + \\ + \frac{1}{2} \int_Y^{\eta(X)} U(\tilde{z}(X + iT))^2 - g(\sigma^*(X + iT))\, dT. \quad (5.32)$$

Letting $X \searrow 0$, the Dominated Convergence Theorem shows that (5.32) holds also for $X = 0$ and all $Y \le Y_c$. Hence $Y \mapsto G(\sigma^*(0 + iY))$ is a C^1 function which satisfies (5.28) on $(-\infty, Y_c]$.

Clearly $\sigma^*(0 + iY) \in \overline{\Theta^-}$ for all $Y \le Y_c$. It remains to investigate the structure of the set $\mathcal{Z}$.

Consider first the case when $Y_c < 0$. Since $\sigma^*(0 + iY_c) = -Y_c \ne 0$, let $(Y_1, Y_c]$ be the maximal interval on which $\sigma^*(0 + iY) \ne 0$ for all $Y \in (Y_1, Y_c]$. We claim that Y_1 is finite. Note that (5.26) holds on $(Y_1, Y_c]$ and, since $U(\tilde{z}(0 + iY)) \in \mathbb{R}$ for all $Y \leq Y_c$, it follows that $\sigma^*(0+iY)$ is real and positive on $(Y_1, Y_c]$. An analysis based on (5.30), similar to the one used for the function $\sigma^*(\pi + iY)$, but using now the fact, which is a consequence of (5.11b), (5.13) and the Cauchy-Riemann equations, that

$$\{U(\tilde{z}(0+iY))\}' < 0 \quad \text{for all} \quad Y \leq Y_c, \tag{5.33}$$

yields that $\sigma^*(0+iY)$ is increasing and concave on $(Y_1, Y_c]$. This shows that necessarily Y_1 is finite, and $\sigma^*(0+iY_1) = 0$. Now (5.28) yields that $\operatorname{Re} G(\sigma^*(0+iY)) < 0$ on $(Y_1 - \varepsilon, Y_1)$ for some $\varepsilon > 0$. Since $\sigma^*(0+iY) \in \overline{\Theta^-}$ for all $Y \leq Y_c$, it follows that $\sigma^*(0+iY) \in \Theta^- \cup \Gamma^-$ on $(Y_1 - \varepsilon, Y_1)$, and therefore $\operatorname{Im} g(\sigma^*(0+iY)) < 0$ on $(Y_1 - \varepsilon, Y_1)$. In the present situation, (5.29) reads as

$$\frac{d}{dY}(\operatorname{Im} G(\sigma^*)) = -\frac{1}{2} \operatorname{Im} g(\sigma^*) \qquad Y \leq Y_c. \tag{5.34}$$

Hence $\operatorname{Im} G(\sigma^*(0+iY)) < 0$ for all $Y < Y_1$, so that $\sigma^*(0+iY) \in \Theta^-$ for all $Y < Y_1$. Then $y_\star \in (-\infty, 0)$ defined to be such that $iy_\star = \tilde{z}(iY_1)$ has the required properties.

Consider now the case when $Y_c = 0$. If there exists $Y_1 < 0$ such that $\sigma^*(0+iY_1)$ is real and non-negative, then (5.34) shows that σ^* is real and non-negative on $(Y_1, 0)$. Therefore, in the z-plane, the function σ is holomorphic in $\widehat{D}$, the intersection of an open disc about zero and the fourth quadrant, and has a continuous extension to the closure of $\widehat{D}$, which is real on the axes. By the Reflection Principle, σ has a holomorphic extension to an open disc centred at zero in the z-plane. Then necessarily $\sigma(x+i0) = -\widetilde{Y}(x+i0)$ for all $x \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, and $x \mapsto \widetilde{Y}(x)$ real-analytic in $(-\varepsilon, \varepsilon)$ implies that $x \mapsto \widetilde{X}(x)$ is real-analytic there, and this contradicts the fact which follows from the Bernoulli condition that

$$\left| \frac{d\widetilde{Z}}{dz}(x+i0) \right| \to \infty \quad \text{as} \quad x \to 0.$$

Therefore $\sigma^*(0+iY)$ is never real for $Y < 0$. It follows from (5.34) that $\operatorname{Im} G(\sigma^*(0+iY)) < 0$ for all $Y < 0$, and hence $\sigma^*(0+iY) \in \Theta^-$ for all $Y < 0$. Hence $y_\star = 0$ has the required properties.

Since $\sigma(iy) \in \Theta^-$ on $(-\infty, y_\star)$, it follows from (5.21) that $0 < \tilde{\phi} < \pi$ on $(-\infty, y_\star)$. If $y_\star < 0$, then $\tilde{\phi} = 0$ on $(y_\star, 0)$ since $\sigma(iy)$ is real and positive there.

This completes the proof of the Theorem.

□

5.3 Properties of Fourier Coefficients

The following result, on the existence of a harmonic continuation of θ to $\Pi_{0,2\pi}$, can now be derived in a similar way to [20] and [21].

Theorem 5.2. *Let θ be given by* (5.5). *Then θ has a harmonic continuation θ^* onto the strip $\Pi_{0,2\pi}$, as an odd function with respect to the line $x = \pi$.*

Moreover, θ^* has a continuous extension to

$$\{(0,y) : y \in \mathbb{R} \setminus \{-y_\star\}\} \cup \{(2\pi, y) : y \in \mathbb{R} \setminus \{-y_\star\}\}, \tag{5.35}$$

with

$$\theta^*(0,y) = \phi(y) \qquad \textit{for} \quad y \in [0,\infty) \setminus \{-y_\star\}\}, \tag{5.36a}$$

$$\theta^*(0,y) = 0 \qquad \textit{for} \quad y \in (-\infty, 0). \tag{5.36b}$$

Here ϕ is defined by $\phi(y) := \tilde{\phi}(-y)$, where $\tilde{\phi}$ is as in Theorem 5.1. Since θ^* is odd with respect to the line $x = \pi$, it follows that

$$\theta^*(\pi, y) = 0 \qquad \textit{for} \quad y \in (-\infty, \infty). \tag{5.36c}$$

Proof of Theorem 5.2. We use the Reflection Principle. Note that

$$2p = \log g(\sigma) \quad \text{for} \quad 0 < x < 2\pi, \ y = 0.$$

Therefore the holomorphic function in $\Pi_{0,2\pi}^-$ given by

$$\theta - ip + \frac{i}{2}\log g(\sigma)$$

is continuous on $\Pi_{0,2\pi}^- \cup \{(x,0) : 0 < x < 2\pi\}$, and real on the real axis. Hence its real part

$$\theta - \frac{1}{2}\arg g(\sigma)$$

has an even harmonic continuation onto the strip $\Pi_{0,2\pi}$. On the other hand, the holomorphic function $\log g(\sigma)$ takes real values on the real axis, and hence its imaginary part has an odd harmonic continuation onto $\Pi_{0,2\pi}$. We conclude that the function

$$\theta^*(x,y) = \begin{cases} \theta(x,y) & \text{for } (x,y) \in \Pi^-_{0,2\pi} \cup \{(x,0) : 0 < x < 2\pi\}, \\ \theta(x,-y) - \arg g(\sigma(x,-y)) & \text{for } (x,y) \in \Pi^+_{0,2\pi}, \end{cases} \tag{5.37}$$

is harmonic in the whole strip $\Pi_{0,2\pi}$. The symmetry property claimed for θ^* in $\Pi_{0,2\pi}$ is straightforward from those of θ and σ in $\Pi^-_{0,2\pi}$. It is clear from (5.37) that θ^* has a continuous extension to the set described in (5.35), and that (5.36a)-(5.36b) hold. This completes the proof. $\square$

Consider now $\theta(x)$, $x \in (0, \pi)$, extended to $\mathbb{R}$ as an odd, 2π-periodic function. Then its Fourier sine coefficients are given by

$$b_n = \frac{2}{\pi} \int_0^\pi \theta(x) \sin nx \, dx, \ \ n \in \mathbb{N}.$$

Theorem 5.3 below is the main result on Fourier coefficients.

Theorem 5.3. *The sequence $\{b_n\}_{n\geq 1}$ is a monotonically decreasing sequence of positive numbers and if*

$$K = \frac{N}{p} + \frac{M}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad K, N, M \in \mathbb{N}, \tag{5.38}$$

then

$$b_K \leq b_N^{1/p} b_M^{1/q} \leq \frac{b_N}{p} + \frac{b_M}{q}. \tag{5.39}$$

In other words, the sequence $\{b_n\}_{n\geq 1}$ is log-convex, and therefore convex and monotonically decreasing to 0.

Proof of Theorem 5.3. We first recall a general result about harmonic functions in strips. If u is bounded in $\Pi_{0,\pi}$, and satisfies the boundary-value problem

$$\begin{aligned} &\Delta u = 0 \quad \text{in} \quad \Pi_{0,\pi}, \\ &u(0,y) = u_0(y), \qquad u(\pi,y) = u_\pi(y), \qquad \text{for all} \quad y \in (0,\infty), \end{aligned}$$

then the following representation formula holds, for all $(x,y) \in \Pi_{0,\pi}$,

$$u(x,y) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{\sin x}{\cosh(y-t) - \cos x}\, u_0(t)\, dt \right.$$
$$\left. + \int_{-\infty}^{\infty} \frac{\sin x}{\cosh(y-t) + \cos x}\, u_\pi(t)\, dt\right). \tag{5.40}$$

The method by which this can be proved consists of relocating the problem from the strip to the unit disc via a conformal map. In the disc the solution is then expressed by Poisson formula. Finally, the solution in the disc is moved back to the strip using the inverse of the previous conformal map. The full details of this strategy are given in [29].

Now recall that the bounded harmonic function θ^* in $\Pi_{0,\pi}$ satisfies the Dirichlet boundary conditions (5.36). By (5.40), we have that

$$\theta(x) = \theta^*(x,0) = \frac{1}{2\pi}\int_0^{\infty} \frac{\sin x}{\cosh t - \cos x}\, \phi(t)\, dt.$$

Since

$$0 \le \frac{\sin x}{\cosh t - \cos x} \le \frac{\sin x}{1 - \cos x} = \cot\frac{x}{2},$$

Fubini's Theorem gives that

$$\begin{aligned}\int_0^{\pi} \theta(x) \sin nx\, dx &= \frac{1}{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\sin x \sin nx}{\cosh t - \cos x}\, \phi(t)\, dt\, dx \\ &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{\pi} \frac{2e^{-t}\sin x \sin nx}{1 + e^{-2t} - 2e^{-t}\cos x}\, dx\, \phi(t)\, dt \\ &= \frac{1}{2}\int_0^{\infty} e^{-nt}\phi(t)\, dt \qquad \text{for all } n \ge 1. \end{aligned} \tag{5.41}$$

To justify that last equality in (5.41), we make the substitution $r := e^{-t}$, $r \in (0,1)$ and $s := x$, so that it remains to prove that

$$\frac{1}{\pi}\int_0^{\pi} \frac{2r \sin s \sin ns}{1 + r^2 - 2r\cos s}\, ds = r^n.$$

But this follows from formula (1.12) which, for the analytic function in the unit disc $\zeta \mapsto i\zeta^n$, expresses the value of its imaginary part at the point $r = re^{i0}$ in terms of the values of its real part on the unit circle.

Since $0 \leq \phi < \pi$ a.e., it follows from (5.41) that the Fourier sine coefficients of θ are positive. If K, N, M, p, q satisfy (5.38) then, by Hölder's Inequality in (5.41), we get

$$\begin{aligned} b_K &= \frac{1}{\pi}\int_0^\infty e^{-Kt}\phi(t)\,dt = \frac{1}{\pi}\int_0^\infty \{e^{-Nt}\phi(t)\}^{1/p}\{e^{-Mt}\phi(t)\}^{1/q}\,dt \\ &\leq \left(\frac{1}{\pi}\int_0^\infty e^{-Nt}\phi(t)\right)^{1/p}\left(\frac{1}{\pi}\int_0^\infty e^{-Mt}\phi(t)\,dt\right)^{1/q} = b_N^{1/p} b_M^{1/q}, \end{aligned}$$

hence the first inequality in (5.39). The second inequality in (5.39) is just Young's Inequality. Since the sequence $\{b_n\}_{n\geq 1}$ is positive, convex and tends to 0 as $n \to \infty$ by Riemann-Lebesgue Lemma, it is well-known (and easy to prove) that it is necessarily monotone. This completes the proof. □

5.4 A Family of Examples

We now give a class of simple examples of nonlinearities g, G which satisfy (5.17)-(5.19). For convenience of notation, in this Section the functions g, G are defined on the complex plane of generic variable $z = x + iy$, rather than $\sigma = \sigma_1 + i\sigma_2$.

Let $G : \mathbb{C} \setminus (-\infty, 0) \to \mathbb{C}$ be given by

$$G(z) = cz^\alpha(z + a_1)(z + a_2)\cdots(z + a_n), \tag{5.42}$$

where $\alpha > 1$, $c > 0$ and $a_1, ..., a_n > 0$, $n \in \mathbb{N}$. Then $g(0) = 0$ and, for all $z \in \mathbb{C} \setminus (-\infty, 0)$,

$$\frac{g(z)}{G(z)} = \frac{\alpha}{z} + \frac{1}{z + a_1} + \cdots + \frac{1}{z + a_n}, \tag{5.43}$$

from where we conclude that

$$\frac{g(z)}{G(z)} \in \mathbb{C}_- \text{ if } z \in \mathbb{C}_+, \quad \text{and} \quad \frac{g(z)}{G(z)} \in \mathbb{C}_+ \text{ if } z \in \mathbb{C}_-. \tag{5.44}$$

In particular g has no zeros in $\mathbb{C} \setminus (-\infty, 0]$.

We study the structure of the set where $\operatorname{Im} G = 0$ and, because of symmetry considerations, it suffices to restrict attention to the upper half-plane. Let

$$\mathcal{A} := \{z \in \mathbb{C}_+ : \operatorname{Im} G = 0\}. \tag{5.45}$$

Since G is real on $\mathcal{A}$, it follows from (5.44) that

$$\operatorname{Im} g \neq 0 \quad \text{on } \mathcal{A}. \tag{5.46}$$

Many of the subsequent arguments are based on the Implicit Function Theorem, which we abbreviate IFT. A first consequence of IFT is that, for every $z \in \mathcal{A}$, there exists a neighbourhood of z whose intersection with $\mathcal{A}$ is a real-analytic curve. In addition, for the function G given by (5.42), one can assert the following.

Lemma 5.4. *Let $z_1, z_2 \in \mathcal{A}$, $z_1 \neq z_2$, be connected by a path Υ contained in $\mathcal{A}$. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then $y_1 \neq y_2$ and, if one assumes that $y_1 < y_2$, then there exists a real-analytic function $\widetilde{\gamma} : [y_1, y_2] \to \mathbb{R}$, with $\widetilde{\gamma}(y_1) = x_1$, $\widetilde{\gamma}(y_2) = x_2$ and*

$$\Upsilon = \{\widetilde{\gamma}(y) + iy : y \in [y_1, y_2]\},$$

Proof of Lemma 5.4. By the IFT, the path Υ is real-analytic. Let us write $\Upsilon = \{u(s) + iv(s) : s \in [a, b]\}$, where u, v are real-analytic and

$$u'(s)^2 + v'(s)^2 > 0 \quad \text{for all } s \in [a, b]. \tag{5.47}$$

Differentiating with respect to s the relation

$$\operatorname{Im} G(u(s), v(s)) = 0 \quad \text{on } [a, b],$$

it follows, taking into account (5.46) and (5.47), that $v'(s) \neq 0$ for all $s \in [a, b]$. This implies the required result. □

We return to the study of the structure of the set $\mathcal{A}$ given by (5.45). Using polar coordinates, so that $z = re^{it}$, $r > 0$, $t \in (0, \pi)$, it follows that

$$z \in \mathcal{A} \quad \Leftrightarrow \quad F(r, t) = 0,$$

where

$$F(r, t) := \operatorname{Im} \left\{ (\cos \alpha t + i \sin \alpha t)(re^{it} + a_1) \cdots (re^{it} + a_n) \right\}.$$

Note that F has a smooth extension to $\mathbb{R} \times (-\pi, \pi)$, and $F(0, \pi/\alpha) = 0$. It follows using the IFT that there exist $\varepsilon, \epsilon > 0$ and a smooth function $\tilde{t} : [0, \varepsilon) \to (0, \pi)$

with $\tilde{t}(0) = \pi/\alpha$, such that, if

$$\Gamma_0^+ := \{re^{i\tilde{t}(r)} : r \in (0,\varepsilon)\},$$

then

$$\mathcal{A} \cap \left\{re^{it} : 0 < r < \varepsilon, \frac{\pi}{\alpha} - \epsilon < t < \frac{\pi}{\alpha} + \epsilon\right\} = \Gamma_0^+.$$

Moreover, it can be shown that

$$\operatorname{Re} G < 0 \quad \text{on} \quad \Gamma_0^+, \tag{5.48}$$

and

$$\operatorname{Im} G(re^{it}) > 0 \quad \text{for all } \ 0 < r < \varepsilon,\ 0 < t < \tilde{t}(r). \tag{5.49}$$

It follows from Lemma 5.4 that there exists $y_0 > 0$ and a real-analytic function $\gamma_0 : (0, y_0) \to \mathbb{R}$ such that

$$\Gamma_0^+ = \{\gamma_0(y) + iy : \ y \in (0, y_0)\}.$$

Since g has no zeros on Γ_0^+, it follows that G is locally injective, and therefore $\operatorname{Re} G$ is a monotone function on Γ_0^+. It follows from (5.48) that

$$\operatorname{Re} G \text{ is decreasing with respect to } y \text{ on } \Gamma_0^+. \tag{5.50}$$

Let $\tilde{y}_0 := y_0/2$, $w_0 := \gamma_0(\tilde{y}_0) + i\tilde{y}_0$, and consider the set

$$\Gamma^+ := \{z \in \mathcal{A} : \ \text{there exists a path } \Upsilon_z \text{ in } \mathcal{A} \text{ joining } z \text{ to } w_0\}. \tag{5.51}$$

The following assertions are immediate consequences of Lemma 5.4 and the IFT: if $z = x + iy$ is such that $0 < y < y_0$, then $\Upsilon_z \subset \Gamma_0^+$; if $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ are such that $y_0 \le y_1 \le y_2$, then $\Upsilon_{z_1} \subseteq \Upsilon_{z_2}$. Let

$$b := \sup\{\operatorname{Im} z : z \in \Gamma^+\}.$$

The preceding considerations show that there exists a function $\gamma : (0, b) \to \mathbb{R}$, which is an extension of γ_0, such that

$$\Gamma^+ = \{\gamma(y) + iy : y \in (0, b)\}.$$

Since Γ^+ is a curve on which G is locally injective, it follows from (5.48) and (5.50) that

$$\operatorname{Re} G < 0 \text{ on } \Gamma^+. \tag{5.52}$$

We now prove that $b = +\infty$. Suppose for a contradiction that b is finite. The definition of b and the IFT show that $\gamma(y)$ does not have any finite limit point as $y \nearrow b$. Therefore either $\lim_{y\to b^-} \gamma(y) = +\infty$, or $\lim_{y\to b^-} \gamma(y) = -\infty$.

Suppose first that $\lim_{y\to b^-} \gamma(y) = +\infty$. It is immediate from (5.42) that in this case $\lim_{y\to b^-} \arg G(\gamma(y) + iy) = 0$, which contradicts (5.52).

Suppose now that $\lim_{y\to b^-} \gamma(y) = -\infty$. Let

$$\Delta^+ := \{x + iy : 0 < y < b, x > \gamma(y)\} \cup \{x + iy : y > b, x \in \mathbb{R}\}.$$

Then, by (5.49) and the Maximum Principle for the bounded harmonic function $\operatorname{Im} 1/G$ in the region $\Delta^+ \cap \{z : |z| > \tilde{t}(\varepsilon)/2\}$, it follows that $\operatorname{Im} G > 0$ in Δ^+. This contradicts the fact, obtained by a suitable application of the IFT, that there exist $a > 0$ sufficiently large and a smooth function $\hat{t} : (a, \infty) \to \mathbb{R}$ with $\lim_{r\to\infty} \hat{t}(r) = \pi/(\alpha + n)$, such that $\mathcal{A}$ contains the arc $\{re^{i\hat{t}(r)} : r > a\}$.

This proves that $b = +\infty$. Hence Γ^+ has the form

$$\Gamma^+ = \{\gamma(y) + iy : 0 < y < \infty\}. \tag{5.53}$$

Let

$$\Theta^+ := \{x + iy : 0 < y < \infty, x > \gamma(y)\}. \tag{5.54}$$

Another application of the Maximum Principle, similar to the preceding one, shows that $\operatorname{Im} G > 0$ in Θ^+. It is obvious how to define Γ and Θ, and it is then easy to verify that (5.17)-(5.19) are satisfied.

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